BIOGRAPHY and CURRICULUM VITAE

David Wing Kay Yeung, PhD, Dr.h.c., FRSS, Senior MIEEE
Director of SRS Consortium for Advanced Study in Cooperative Dynamic Games
Hong Kong Shue Yan University
and
Co-director of Center of Game Theory
Faculty of Applied Mathematics – Control Processes
Saint Petersburg State University
and
Joint Distinguished Professor
Department of Finance
Asia University

Biography

Prof. Dr. Dr.h.c. David W. K. Yeung is Director of SRS Consortium for Advanced Study in Cooperative Dynamic Games at Shue Yan University and Co-director of Centre of Game Theory at Saint Petersburg State University, Distinguished Adjunct Professor of Asia University, He is also Distinguished Honorary Professor of Qingdao University - an honor he received along with Nobel laureates John Nash, Reinhard Selten, Robert Aumann and Lloyd Shapley. The SRS Consortium for Advanced Study in Dynamic Cooperative Games is a joint research initiative under the auspices of Saint Petersburg State University, Russian Academy of Sciences and Shue Yan University. He has held Faculty positions at Queen's University, University of Hong Kong and National University of Singapore.

Yeung obtained his BSocSc in economics and statistics from University of Hong Kong and his PhD in mathematical economics from York University. He studied in the Doctor of Sciences (Doktor Nauk – Continental European higher doctorate above PhD) program in applied mathematics at Saint Petersburg State University and was awarded the University's highest degree of Dr.h.c. for his outstanding contributions in differential games. The Doktor Nauk program requires an average of 15 years to complete and the development of a new scientific field (in contrast with the PhD program which requires the development of a new topic).

Yeung's main areas of research are game theory, optimization and stochastic processes. He serves as Managing Editor of International Game Theory Review, Guest Editor of Annals of Operations Research, Associate Editor of Dynamic Games & Applications, Associate Editor of Operations Research Letters, Editor of the Routledge (Taylor & Francis) Series on Economics and Optimization, and Board of Editors of Mathematical Game Theory & Applications. Yeung has published extensively in world renowned science journals, in particular, Automatica, Journal of Optimization Theory and Applications, Annals of Operations Research, European Journal of Operational Research, International Game Theory Review and Mathematical Biosciences.

Among Yeung's ground-breaking academic work include (i) the origination of the notion of subgame consistency and a generalized theorem for the derivation of analytically tractable subgame consistent solutions which made possible the rigorous study of dynamic stochastic cooperation; and (ii) a new paradigm of games -- 'durable-strategies dynamic games' which provides a new perspective approach with a new class of strategies in dynamic game theory. The addition of durable strategies in the strategies set of dynamic games does not only make game theory applications possible for analysis of real-life problems with durable strategies but also establish a new theoretical framework with novel optimization theory.

Other pioneering academic work accredited to Yeung include – random horizon subgame consistent cooperative solution, randomly furcating differential games, Yeung's Condition on irrational behaviour proof, stochastic differential games with overlapping generations of uncertain types of players, dynamically consistent solution for cooperative games with asynchronous players' horizons, solution theorem for feedback Nash equilibria in endogenous horizons differential games, Yeung's Lemma which demonstrates the correspondence of degenerate and non-degenerate FNE, the Yeung Number which identifies the number of embedded coalitions, discrete-time random-horizon Bellman equation and HJB equations, stochastic dynamic Slutsky equations, inter-temporal Roy's identities, the Lotka-Volterra-Yeung density function and dynamic Nash bargaining.

In management sciences, Yeung's pioneering work include subgame consistent schemes for global pollution management, dynamic games in management science with interest rate uncertainty, analysis on institutional investor speculation in stochastic differential games framework, time-consistent corporate joint ventures and subgame consistent cooperative solution mechanism for randomly furcating stochastic differential games to handle complex uncertainties in business syndicates. In economics, Yeung extended the Saint Petersburg tradition led by Leonid Kantorovich's Nobel Prize winning work in optimal resource allocation to multiple-agent subgame consistent economic optimization.

Yeung and Leon Petrosyan co-authored the world's first book on Cooperative Stochastic Differential Games. In the book's advanced praise George Leitmann (Berkeley) stated that this book will be the 'bible' of the field for years to come. Vladimir Mazalov (Russian Academy of Sciences) praised the book's advancements in dynamic stability as a continuation of the great tradition of A. Lyapunov, L. S. Pontryagin and V. Zubov.

Yeung and Petrosyan co-authored the world's first treatise on Subgame Consistent Economic Optimization. Vladimir Mazalov praised the text as a Russian classic in mathematics and economics and it expanded L.V. Kantorovich's award-winning work in economic optimization significantly in the new directions of game-theoretic interaction, dynamic evolution, stochasticity and subgame consistency.

Yeung also authored a premier treatise on dynamic consumer theory with stochastic dynamic Slutsky equations. This book is the world's first text on consumer theory in a stochastic dynamic framework. It expands the static consumer theory into stochastic dynamic consumer theory. Prominent highlights of the book include a series of stochastic dynamic Slutsky equations and inter-temporal Roy's identity. *"The stochastic dynamic Slutsky equations in this book formed a new milestone in consumer theory after the classic work of Slutsky."* (Vladimir Mazalov, Russian Academy of Sciences)

Yeung and Leon Petrosyan co-authored the world's first book on Subgame Consistent Cooperation, which is a comprehensive treatise on the concepts and solution mechanisms of their pioneering work on cooperative subgame consistency. In the book's back-cover praise Vladimir Mazalov stated, "*The 2004 Nobel Economics Prize was given to works in economic policies under the concept of time consistency with mathematical construction less general, rigorous and precise than that later developed in this book. .... In terms of advancement in practical applications this book is highly important theoretically and technically on top of economic interpretation*."

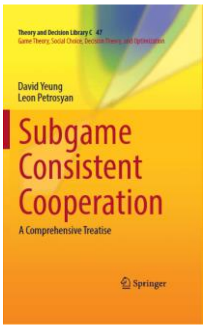
Yeung and Leon Petrosyan also co-authored another world's first text on Dynamic Shapley Value and Dynamic Nash Bargaining. Vladimir Turetsky stated, "*the contribution of the book is the dynamic extension of two milestone results in Cooperative Games, the bargaining solution of Nash and the value of Shapley. Real-life processes in economics, biology, social life, etc. are inherently* *dynamic,…….. . Therefore, I believe that mathematical results presented in the book will have a long echo in various applications in different and unexpected fields."*

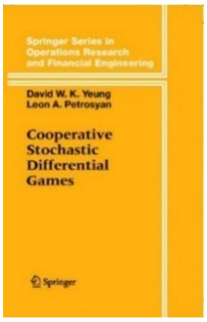
Yeung and Leon Petrosyan developed a novel dynamic game theory paradigm – "Durable-strategies Dynamic Games". It yields a new system of dynamic games in which the players' strategies involve durable controls. In addition, a new dynamic optimization technique is also established. This paradigm fills a significant gap in conventional dynamic game theory. The establishment of durable-strategies dynamic games provides an augmentation of dynamic game theory with durable strategies in decision-making schemes. Praises for the book from eminent dynamic game theorists stated that:

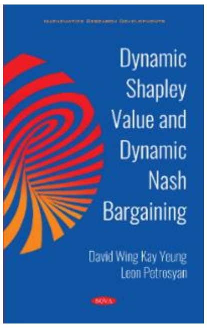
"*Not too many new concepts have been introduced in dynamic games since their inception. The introduction of the concept of durable strategies changes this trend and yields important contributions to environmental and business applications*." -- Dušan M Stipanović.
"*Before this book, the field simply did not realize that most of our strategies are durable and entail profound effects in the future. Putting them into the mathematical framework of dynamic games is a great innovative effort*." -- Vladimir Turetsky.

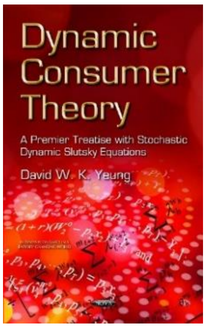
"*Durable-strategies Dynamic Games is truly a world-leading addition to the field of dynamic games. It is a much needed publication to tackle increasingly crucial problems under the reality of durable strategies*." -- Vladimir Mazalov.

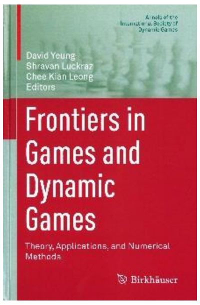
Books Published:

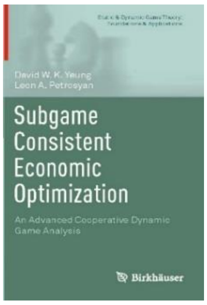


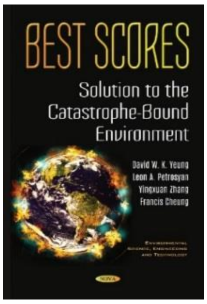


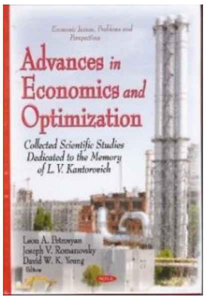


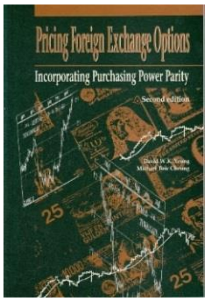


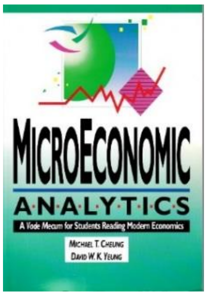


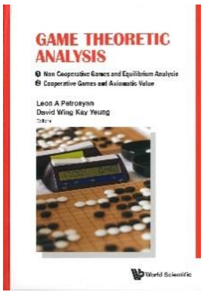


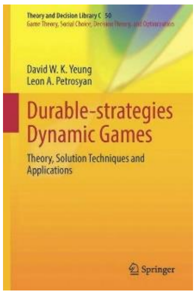












CURRICULUM VITAE

David Wing Kay Yeung, PhD, Dr.h.c., FRSS, Senior MIEEE

Director of SRS Consortium for Advanced Study in Cooperative Dynamic Games
Hong Kong Shue Yan University
and
Co-director of Center of Game Theory
Faculty of Applied Mathematics – Control Processes
Saint Petersburg State University
and
Distinguished Adjunct Professor
Department of Finance
Asia University

PROFESSIONAL APPOINTMENTS

Distinguished Guest Professor, Qingdao University
Distinguished Honorary Professor, Qingdao University
Managing Editor of International Game Theory Review
Guest Editor – Annals of Operations Research
Guest Editor – Annals of the International Society of Dynamic Games
Associate Editor of Dynamic Games and Applications
Associate Editor of Operations Research Letters
Series Editor of Routledge Series on Economics and Optimization
Editorial Member – Mathematical Game Theory and Applications
Editorial Member – Saint Petersburg Vijestnik of Applied Mathematics, Informatics, Control
Executive Committee – International Society of Dynamic Games
Mathematical Reviews' Reviewer – American Mathematical Society
Professorial Associate – Victoria University
Advisory Board Member – Institute of Advanced Management Development, HKMA
Honorary Director - Centre for Corporate Management, HKMA
Senior MIEEE – Institute of Electronic and Electrical Engineering
Fellow of the Royal Statistical Society
Founding Advisor, International Society of Dynamic Games – China Chapter
Guest Editor – Journl of the Operations Research Society of China

ADDRESS
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Hong Kong Shue Yan University
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E-mail: dwkyeung@hksyu.edu
and
Centre for Game Theory, Faculty of Applied Mathematics & Control Processes
Saint Petersburg State University
FAX/TEL: (7) 812 350 5067 E-mail: spbuoasis7@peterlink.ru

WORKING EXPERIENCE

Jan. 2010-	Director of SRS Consortium for Advanced Study in Cooperative Dynamic Games, Shue Yan University
Feb. 2000 -	Co-Director, Centre for Game Theory Faculty of Applied Mathematics and Control Processes Saint Petersburg State University
Aug. 2019 -	Distinguished Adjunct Professor, Asia University
Sept. 2018-	Professorial Associate, Victoria University
Jan. 2010 – Dec. 2021	Distinguished Research Professor, Shue Yan University
Sep. 2020 – Dec. 2021	Head of Department of Economics and Finance, SYU
Jan. 2010-Aug. 2020	Head of Department of Business Administration, SYU
Jan. 2022 -	Emeritus Distinguished Research Professor, SYU
July 2006 – Jan. 2010	Chair Professor in Game Theory Department of Finance and Decision Sciences Hong Kong Baptist University
June 2004 – Dec. 2020	Kantorovich Research Chair in Stochastic Differential Games Faculty of Applied Mathematics and Control Processes Saint Petersburg State University
Jan. 1999 –June 2006	Professor, Department of Finance and Decision Sciences Hong Kong Baptist University
Feb. 2000 – Jan. 2010	Director, Centre of Game Theory Hong Kong Baptist University
May - Jun 2000; May-Jul 2001; Mar, Apr 2002	Acting Head, Department of Finance and Decision Sciences Hong Kong Baptist University
July 1999 –Dec. 2000	Court Member – Hong Kong Baptist University
Dec. 2001 -	Distinguished Guest Professor, Qingdao University
Aug. 2002 -	Distinguished Honorary Professor, Qingdao University
April 2006-	Honorary Research Fellow – Center of Urban Planning & Environmental Management, the University of Hong Kong
April 2006-	Research Associate – Center of Strategic Economic Study, Victoria University, Australia
Feb. 2008-	Honorary Fellow – Center for Suicide Research and Prevention, The University of Hong Kong
Jan. 2002 –Dec 2005	Honorary Professor, Faculty of Social Sciences, The University of Hong Kong
Sept. 1997 – Aug. 1999	Associate Professor School of Economics and Finance

	The University of Hong Kong
Jan. 1996 – Aug. 1997	Senior Lecturer, Department of Economics and Statistics
	The National University of Singapore
Sept. 1989 – Dec. 1995	Lecturer, School of Economics and Finance
	Fellow, Centre of Urban Planning & Environment Management
	The University of Hong Kong
July 1987 – Sept. 1989	Associate Professor, Department of Economics
	University of Windsor
July 1986 – June 1987	Lecturer, Department of Economics
	York University, Department of Economics
July 1984 - June 1986	Visiting Assistant Professor, Department of Economics
	Queen's University

EDUCATION AND DEGREES

B.Soc.Sc. (Economics & Statistics) University of Hong Kong
M.A. (Economics) York University
Ph.D. (Economics) York University
Supervisors: Charles Plourde, John Buttrick, Stanley Warner
Dr.h.c. (Applied Mathematics) Saint Petersburg University
Studied in the Doktor Nauk Program (Applied Mathematics - higher doctorate above PhD) at Saint Petersburg State University and was awarded the highest degree Dr.h.c.
Supervisor: Leon Petrosyan

SCHOLARSHIPS

1983-1984	Doctoral Fellowship
	Social Sciences and Humanities Research Council of Canada
1982-1983	Doctoral Fellowship
	Social Sciences and Humanities Research Council of Canada
1981-1982	York Graduate Scholarship
1973	Grantham Scholarship

Scholarships awarded but not taken up due to concurrent holding of other fellowships/scholarships

1982	Doctoral Scholarship
	Canada Mortgage and Housing Corporation
1982	Ontario Graduate Scholarship
1981	Queen’s Doctoral Scholarship

HONORS AND AWARDS

Saint Petersburg State University’s highest degree, Dr.h.c., for outstanding contributions in differential game theory (while studying D.Sc. (Doktor Nauk) in Applied Mathematics)

Distinguished Honorary Professor, Qingdao University – appointed along with Nobel Laureates John Nash, Reinhard Selten, Robert Aumann and Lloyd Shapley.

Plenary Speaker at the International Congress of Mathematicians ICM 2002 historic first Satellite Conference on Game Theory and Applications – other plenary speakers included Nobel Laureates John Nash, Reinhard Selten, Robert Aumann and Lloyd Shapley.

Kantorovich Research Chair in Stochastic Differential Games, Saint Petersburg State University – for signal research contributions in stochastic differential games.

Invited Nominator for the Nobel Memorial Prize in Economic Sciences.

FIELDS OF SPECIALIZATION AND RESEARCH INTERESTS:

Game Theory, Operations Research, Stochastic Processes, Stochastic Control, Probability Theory, Financial Engineering and Modelling, Mathematical Economics, Microeconomics, Resource Economics, Environmental Economics, Economics of E-Commerce.

PROFESSIONAL QUALIFICATION

FAIBA – Fellow of Accredited Institute of Business Administrants

PROFESSIONAL AND ACADEMIC ACTIVITIES

Academic Journal Editorship

Managing Editor -	International Game Theory Review
Guest Editor -	Annals of Operations Research
Guest Editor -	Annals of the International Society of Dynamic Games
Guest Editor -	Journal of the Operations Research Society of China
Guest Editor -	Journal of Computational and Applied Mathematics
Associate Editor -	Dynamic Games and Applications
Associate Editor-	Operations Research Letters
Series Editor-	Routledge Series on Economics and Optimization
Editorial Board Member -	Mathematical Game Theory and Applications
Commissioned Reviewer -	Mathematical Reviews (American Mathematical Society)
Editorial Board Member -	Vestnik of Saint Petersburg University: Applied Mathematics, Computer Science, Control Processes

Books and Edited Volumes Referee – Birkhauser, Prentice Hall, John-Wiley, Springer-Verlag, Kluwer Science Publishers, World Scientific Publishers

Journals Referee - American Economic Review
Automatica
American Journal of Agricultural Economics
Annals of Operations Research
Annals of the International Society on Dynamic Games
Canadian Journal of Economics
Corporate Governance
Discrete Dynamics in Nature and Society
Dynamic Games and Applications
European Journal of Operational Research
Hong Kong Economic Papers
IEEE Transactions on Automatic Control
International Game Theory Review
Journal of Computational and Applied Mathematics
Journal of Economic Dynamics and Control
Journal of Environmental Economics and Management
Journal of Mathematical Analysis and Applications
Journal of Mathematical Economics
Journal of Optimization Theory and Applications
Journal of the Royal Society Interface
Marketing Science
Mathematical Methods of Operations Research
Natural Resource Modeling
Omega
Operations Research Letters
Optimal Control, Method and Applications
Optimization Letters

Physica A
Quarterly Journal of Economics
Singapore Economic Review

Other Professional and Academic Activities

Founding Advisor - International Society of Dynamic Games –China Chapter
Executive Committee- International Society of Dynamic Games
Honorary Adviser - The Institute of Business Administrants (IBA)
Editorial Board Member Mathematical Game Theory and Applications
Guest Editor Annals of Operations Research
Editorial Advisory Council - Pacific Economic Review
Associate Editor - Hong Kong Economic Papers
Department Representative - Canadian Operational Research Society
Staff Member - Great Lakes Institute, Ontario, Canada
Reviewer - Mathematical Reviews (American Mathematical Society)
Executive Committee - Hong Kong Economic Association
Advisor - Victoria Jaycees, The Hong Kong Junior Chamber of Commerce & Jaycees International
Member - China-Hongkong Securities Markets Research Team, HKSE
Reviewer- National Sciences and Engineering Research Council of Canada (NSERC)
Program Committee- 1999 Far Eastern Meeting of the Econometric Society
External Examiner- Graduate (Economics) Programme, Chinese University of Hong Kong
Program Committee- The Far Eastern Meeting of the Econometrics Society 1999, Singapore, July 1999.
International Program Committee- Ninth International Symposium on Dynamic Games and Applications, Adelaide, Dec 2000.
International Program Committee- International Federation of Automatic Control – 11th Workshop on Control Application of Optimization, St Petersburg, July 2000.
Program Committee- JSAI 2001 International Workshop on Agent-based Approaches in Economics and Social Complex System, Matsue City, Japan, May 2001.
Program Committee- International Conference on Logic, Game Theory and Social Choice, St Petersburg, June 2001.
International Program Committee - 4th International Conference on Computational Intelligence and Multimedia Applications, Yokosuka, Japan 2001.
International Program Committee- Tenth International Symposium on Dynamic Games and Applications, St Petersburg, July, 2002.
Program Committee- Second International Workshop on Agent-based Approaches in Economics and Social Complex System, Tokyo, Japan, 2002.
Vice Chairperson
Academic Program Committee – International Congress of Mathematicians 2002 Satellite Conference on Game Theory and Applications
Program Committee- International Conference on Logic, Game Theory and Social Choice 3, Siena, September 2003.

Program Committee-	Third International Workshop on Agent-based Approaches in Economics and Social Complex System, Kyoto, Japan, May 2004.
International Program Committee-	Eleventh International Symposium on Dynamic Games and Applications, Tucson, December, 2004.
Technical Program Committee-	The 7th Australian Joint Conference on Complex System, Cairns, Australia, December 2004.
Program Co-chairperson-	First Workshop of the Chinese Chapter of the International Society of Dynamic Games, Qingdao, China, August 2004.
Program Committee	International Conference on Stability and Control Processes -- dedicated to V. I. Zubov, St Petersburg, Russia, Jun/Jul 2005.
Scientific Committee	6th Meeting on Game Theory and Practice, Zaragoza, Spain, 10-12 July 2006.
International Program Committee-	Twelve International Symposium on Dynamic Games and Applications, Sophia Antipolis, France, July, 2006.
Program Committee	The First World Congress on Social Simulation, Kyoto, Japan, August 2006.
International Program Committee-	Twelve International Symposium on Dynamic Games and Applications, Sophia-Antipolis, France, July, 2006.
Program Committee	Fifth International Workshop on Agent-based Approaches in Economics and Social Complex System, Tokyo, August, 2007.
International Program Committee-	First International Conference on Game Theory and Management, St Petersburg, Russia, June 2007.
International Program Committee-	Second International Conference on Game Theory and Management, St Petersburg, Russia, June 2008.
International Program Committee-	Thirteenth International Symposium on Dynamic Games and Applications, Wroclaw, Poland, June-July 2008.
Program Committee	World Congress on Social Simulation 2008, Fairfax, USA, July, 2008.
International Program Committee-	Third International Conference on Game Theory and Management, St Petersburg, Russia, June 2009.
International Program Committee-	Fourth International Conference on Game Theory and Management, St Petersburg, Russia, June 2010.
International Program Committee-	Fifth International Conference on Game Theory and Management, St Petersburg, Russia, June 2011.
International Program Committee-	Sixth International Conference on Game Theory and Management, St Petersburg, Russia, June 2012.
International Program Committee-	Seventh International Conference on Game Theory and

Management, St Petersburg, Russia, June 2013.

International Program Committee- Eighth International Conference on Game Theory and Management, St Petersburg, Russia, June 2014.

International Program Committee- Tenth International Conference on Game Theory and Management, St Petersburg, Russia, June 2016.

International Program Committee- Eleventh International Conference on Game Theory and Management, St Petersburg, Russia, June 2017.

Honorary Program Chair International Society of Dynamic Games (ISDG)-China Chapter Conference on Dynamic Games and Game Theoretic Analysis, NingBo, China, 3-5 August 2017.

International Program Committee- Twelfth International Conference on Game Theory and Management, St Petersburg, Russia, June 2018.

International Program Committee- Thirteenth International Conference on Game Theory and Management, St Petersburg, Russia, June 2019.

International Program Committee- Fourteenth International Conference on Game Theory and Management, St Petersburg, Russia, June 2020.

Advisory Committee- International Conference on The Evolution of Digital Entrepreneurship, FinTech and FinReg, Hong Kong, March, 2021.

International Program Committee- Fifteenth International Conference on Game Theory and Management, St Petersburg, Russia, June 2021.

International Scientific Committee- International Congress of Mathematician – Satelite Confernce on Game Theory and Application, St Petersburg, June, 2022.

International Scientific Committee- Sixteenth International Conference of Game Theory and Management, St Petersburg, Russia, June, 2023.

GRANTS

RGC Competitive Earmarked Research Grant (CERG) 1999-2004
Project Title - Investor Behavior in Equity Markets: A Stochastic Differential Game Analysis.
HK$ 477,000, (Sole investigator).

FRG Grants (HKBU) 2000-2001
Project Title - Stock Markets and the World's Major Financial Centers,
HK$ 57,500 (Principal investigator).

FRG Grants (HKBU) 2003-2004
Project Title - Stochastic Dynamic Investment Games: Towards a New Paradigm in Financial Analysis,
HK$66,840 (Sole investigator).

FRG Grants (HKBU) 2004-2005

Project Title - Stable Corporate Joint Ventures: The Dynamic Shapley Approach, HK$146,600 (Sole investigator).

RGC Competitive Earmarked Research Grant (CERG) 2004-2007
Project Title – Consistent Economic Cooperation in Stochastic Dynamic Environment, HK$339,150 (Principal investigator).

FRG Grants (HKBU) 2005-2007
Project Title – Subgame-consistent Cooperative Nonrenewable Resource Management Under Asynchronous Planning Horizon,
HK$73,800 (Principal investigator).

FRG Grants (HKBU) 2007-2008
Project Title – Irrational-Behaviour-Proof Solutions in Dynamics Cooperative Schemes
HK$93,160 (Principal investigator).

Strategic Development Fund (HKBU) 2007-2009
Project Title – Consortium for Advanced Study in Dynamic Cooperative Game,
HK$255,253 (Sole investigator).

European Union Research Grant 2007-2009
Project Title – Technology-Oriented Cooperation and Strategies in India and China: Reinforcing the EU dialogue with Developing Countries on Climate Change Mitigation.
With Partners: University of Cambridge, Ecole Polytechnique, Tsinghua University, India Institute of Management, FEEM, KANLO and ORDECSYS,
HK$11,308,579 (Euros 1,069,000).

RGC Competitive Earmarked Research Grant (CERG) 2007-2010
Project Title: Subgame-consistent Solutions for Asynchronous-horizon Cooperative Stochastic Differential Games,
HK$445,000 (Principal investigator).

Shue Yan University Research Grant 2011-2012
Project Title -- Time Consistent Management Schemes for Collaborative Business Development
Project Investigators: David Wing-kay YEUNG, Wing-Fu SZETO, Che-fai LAM
HK$24,000

Shue Yan University Publication Grant 2011-2012
Book Title: Subgame Consistent Economic Optimization: An Advanced Cooperative Dynamic Game Analysis.
Sole Investigators
HK$10,000

RGC Institutional Development Scheme 2017-2020
Project Title: Further Enhancement of Interdisciplinary Research at Hong Kong Shue Yan University through the Establishment of the Centre of Interdisciplinary Research in Evidence-Based Practice
Team Leader
HK$6,221,500

SCHOLARLY WORK

PUBLICATIONS
Refereed Articles and Refereed Books

1. Petrosyan, L.A., D.W.K. Yeung, and E.M. Parilina: Mathematical Game Theory at St. Petersburg State University, International Game Theory Review, 2024, https://doi.org/10.1142/S0219198923500196
2. David W. K. Yeung and Leon A. Petrosyan (2024) Subgame-Consistent Cooperative Equilibria of Multi-Objective Dynamic Games, International Game Theory Review, 2024, https://doi.org/10.1142/S0219198923500202
3. Petrosyan, L.A., Yeung, D.W.K. and Pankratova Y.: Characteristic Functions in Cooperative Differential Games on Networks, *Journal of Dynamics and Games*, 2023. doi: 10.3934/jdg.2023017
4. Yeung, D.W.K. Petrosyan, L.A. and Zhang, Y.X.: A Dynamic Network Game of the FinTech Industry, Journal of the Operations Reseach Society of China, 2023, https://doi.org/10.1007/s40305-022-00434-4
5. Yeung, D.W.K. and Zhang, Y.X.: Bi-objective Optimization – A Pareto Method with Analytical Solutions, *Applied Mathematics*, Vol.14, No.1, pp.57-81, 2023, doi: 10.4236/am.2023.141004.
6. Wong, W.K., Yeung D.W.K. and Lu R.: The Mean-variance Rule for Investors with Reverse S-shaped utility, forthcoming in *Annals of Financial Economics*, 2023.
7. Yeung, D.W.K. and Petrosyan, L.A.: Durable-strategies Dynamic Games: Theory, Solution Techniques and Applications, Springer, Switzerland, 303pp, 2022, ISBN 9783030927417.
8. Yeung, D.W.K. and Wong, W.K.: An Informational Theory of the Dynamic Value of the Firm, *Annals of Financial Economics*, 2022. https://doi.org/10.1142/S2010495222500166.
9. Yeung, D.W.K. and Petrosyan, L.A.: Generalized dynamic games with durable strategies under uncertain planning horizon, *Journal of Computational and Applied Mathematics*, 395 (2021) 113595. https://doi.org/10.1016/j.cam.2021.113595
10. Yeung, DWK, and Petrosyan, LA, (2021). Asynchronous Horizons Durable-Strategies Dynamic Games and Tragedy of Cross-Generational Environmental Commons, *International Game Theory Review*. Available at: doi. org/10.1142/S0219198921500201.
11. Yeung, D.W.K., Petrosyan, L.A. and Zhang, Y.X.: Trade with Technology Spillover: A Dynamic Network Game Analysis, *International Game Theory Review*, 2021, Vol. 23, No.1, 205011 (1-31).
12. Yeung, D.W.K., Zhang, Y.X., Bai, H.T. and Islam, S.: Collaborative Environmental Management for Transboundary Air Pollution Problems: A Differential Levies Game, *Journal of Industrial and Management Optimization*, 2021, 17 (2) : 517-531. Doi:10.3934/jimo.2019121.
13. Petrosyan L., Yeung D., Pankratova Y. (2021) Dynamic Cooperative Games on Networks. In: Strekalovsky A., Kochetov Y., Gruzdeva T., Orlov A. (eds) Mathematical Optimization Theory and Operations Research: Recent Trends. MOTOR 2021. Communications in Computer and Information Science, vol 1476. Springer, Cham. https://doi.org/10.1007/978-3-030-86433-0_28
14. Petrosyan, L.A. and Yeung, D.W.K.: Shapley Value for Differential Network Games: Theory and Application, *Journal of Dynamics and Games*, 2021, 8(2): 151-166 doi: 10.3934/jdg.2020021.
15. L. Petrosyan, D. Yeung, and Y. Pankratova (2021): differential nonzero-sum game on

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RESEARCH IN PROGRESS

Research papers in Progress:
Dynamic Cooperation in Pollution Reduction
Dynamic Stability of Solutions in Cooperative Differential Games: A Critical Survey
The Cooperative Game of Public Healthcare Provision
Subgame Consistent Cooperation in Stochastic Dynamic Resource use
A Dynamic Interactive Model of Viruses and Deleterious Immunity Response
A Subgame-consistent Approach to Asian Pacific Economic Cooperation
On the Crux of Successful Economic Cooperation in the Pan Pearl-River Delta
Game-theoretic Analysis of CEPA

Books and Monographs in Progress:
Elephants in the Pond: Institutional Investors and Financial Games
Advanced Dynamic Decision Sciences: Selected Scientific Work of David W. K. Yeung

The Development of Economic Thoughts from a Mathematician's Perspective
Cooperative Environmental Economics

PLENARY LECTURES, INVITED PAPERS & CONFERENCE PRESENTATIONS

"A Large-scale Network Differential Games with the Petrosyan Cooperative-trajectory Characteristic Function", Plenary Lecture presented at Workshop on Dynamic Games and Applications in Honor of Prof. Leon A. Petrosyan, Tashkent, Uzbekistan October 3–5, 2023.

"Multi-objective Dynamic Games", Plenary Lecture presented at the Sixteenth International Conference of Game Theory and Management, St Petersburg, 28-30 June, 2023.

"The mean-variance rule for investors with reverse S-shaped utility", Keynote address presented at the Virtual IIF International Research Conference & Award Summit, (with W.K. Wong and R. Lu), 6-8 January, 2023.

"Durable Strategies Dynamic Games under Uncertainty", Plenary Lecture presented at 2022 International Congress of Mathematician (ICM) Satelite Conference on Game Theory and Applications, St Petersburg, 28 June - 1 July, 2022.

"Cooperative Network Games with Partner Sets", (with L. Petrosyan and Y. Pankratova), paper presented at 2022 International Congress of Mathematician (ICM) Satelite Conference on Game Theory and Applications, St Petersburg, 28 June - 1 July, 2022.

"Differential Games on Networks", (with L. Petrosyan and Y. Pankratova), paper presented at the International Conference Devoted to the 100^{th} Anniversary of Academician E.V.Mischenko, Moscow, 8-9 June 2022.

"Advances in Durable Strategies Dynamic Games", Plenary Lecture presented at the 15th International Conference on Game Theory and Management (GTM 2021), St Petersburg, 23-25 June 2021.

"The Economics Behind the Evolution of the FinTech Industry" (with T. Yuen and YX Zhang) presented at International Conference on The Evolution of Digital Entrepreneurship, FinTech and FinReg, Hong Kong, March, 2021.

"A Dynamic Network Game of Trade Augmentation under Technology Spillover" (with Y.X. Zhang), paper presented at the 31 Stony Brook International Conference on Game Theory - Workshop on Differential Games, Stony Brook, New York, 27-28 July 2020.

"Construction of Dynamically Stable Solutions in Network Differential Games" (with L.A. Petrosyan) paper presented at the International Conference on "Stability, Control, Differential Games" devoted to the 95th anniversary of Academician N.N. Krasovskii (SCDG2019), Yekaterinburg, Russia, 12-15 September 2019.

"Brexit: A Cross-disciplinary Study in Politics, Socio-Economics and Game Theory with Predicted Impacts for Evidence Verification," (with A.W.C. Lui) paper presented at The Institutional Development Scheme International Conference -- Bridging The Gap: Translating Interdisciplinary Research into Evidence-based Practice, Hong Kong, 3-4 May 2019.

"Bi-level Cooperation in a Class of n-person Differential Games", (with L.A. Petrosyan) paper presented at the 17th IFAC Workshop on Control Applications of Optimization, Yekaterinburg, Russia, October 15–19, 2018.

"Dynamically Consistent Cooperative Solution of a Dynamic Game with Coalitional Blocs," (with L.A. Petrosyan), paper presented at the 18^{th} *International Symposium on Dynamic Games and Applications*, Grenoble, France, July 9-12, 2018.

"Time-Consistency, Subgame Consistency, Subgame Perfection and Dynamic Stability of Solutions in Dynamic Games", (with L.A. Petrosyan), paper presented at the 18^{th} *International Symposium on Dynamic Games and Applications*, Grenoble, France, July 9-12, 2018.

"Solutions of cooperative Dynamic Games with Control Lags", (with L.A. Petrosyan) paper presented at the *Twelfth International Conference on Game Theory and Management (GTM2018)*, St. Petersburg, Russia, June 27-29, 2018.

"Transnational Co-development of GreenTech and Integrated Global Response to Anthropogenic Environmental Impacts", paper prepared for presentation at *Anthropocene Conference 2018*, Hong Kong, 29 May – 1 June 2018.

"Dynamically Stable Provision of Public Goods: A NTU Cooperative Differential Game Approach", Plenary Lecture presented at the *ISDG-China Chapter International Conference on Dynamic Games and Game Theoretic Analysis*, NingBo, 3-5 August 2017.

"Cooperative Stochastic Differential Games with Information Adaptation" (with O. Petrosyan), paper presented at the *International Conference on Communication and Electronic Information Engineering*, Guangzhou, October, 2016.

"On Subgame Consistent Solution for NTU Cooperative Stochastic Dynamic Games" (with L.A. Petrosyan), paper presented at *European Meeting on Game Theory (SING11-GTM2015)*, St Petersburg, July 8-10, 2015.

"Strategically Supported Time-consistent Solutions in NTU Differential Games" (with L. A. Petrosyan) paper presented at the *Sixteenth International Symposium on Dynamic Games and Applications*, Amsterdam, the Netherlands, 9-12 July 2014.

"Subgame-consistent Cooperative Solution of Stochastic Dynamic Game of Public Goods Provision", paper presented at the *Seventh International Conference on Game Theory and Management*, St Petersburg, Russia, June 26-28, 2013.

"Subgame-consistent Solutions for Cooperative Stochastic Dynamic Games with Randomly Furcating Payoffs", paper presented at the Fourth World Congress of the Game Theory Society, Istanbul, Turkey, 23-26 July 2012.

"Subgame Consistent Solution for a Cooperative Differential Game of Climate Change Control", (with L.A. Petrosyan) paper presented at the *Sixth International Conference on Game Theory and Management,* St Petersburg, Russia, June 27-29, 2012.

"Subgame Consistent Cooperative Solutions in Stochastic Differential Games with Asynchronous Horizons and Uncertain Types of Players", paper presented at the *Sixth International Conference on Game Theory and Management,* St Petersburg, Russia, June 27-29, 2012.

"Dynamically Stable Vertical Integration of Firms in a Supply Chain", International Conference on Industrial Engineering and Operations Management, Kuala Lumpur, Malaysia, January 22 – 24, 2011.

"Subgame Consistent Solutions for Random Horizon Cooperative Dynamic Games", paper presented at the *Fifth International Conference on Game Theory and Management,* St Petersburg, Russia, June 26-28, 2011.

"Dynamically Consistent Solution for a Cooperative Differential Game of Climate Change Control", paper presented at *the Fourteenth International Symposium on Dynamic Games and Applications*, Banff, Canada, June 19 - 23, 2010.

"The Detailization of the Irrational Behavior Proof Condition for the Emission Reduction Game," (with L.A. Petrosyan and A.V. Iljina) paper presented at the *First Conference of Chinese Game Theory and Experimental Economics Association,* Beijing, China, August 24-26, 2010.

"The Detailization of The Irrational Behaviour Proofness Condition," (with L.A. Petrosyan and V.V. Jhuk) paper presented at the *Third International Conference on Game Theory and Management*, St Petersburg, Russia, June 24-26, 2009.

"On Financial Constraint in Intertemporal Cooperative Pollution Management Games" (with Y.X. Zhang), paper presented at *TOCSIN Cambridge Meeting,* Cambridge, U.K., 10-11 December 2008.

"Cooperative Game-theoretic Mechanism Design for Optimal Resource Use," Plenary Lecture delivered at the *Second International Conference on Game Theory and Management*, St Petersburg, Russia, June 26-27, 2008 (Nobel Laureate John Nash was another Plenary Speaker).

"Dynamically Consistent Cooperative Solutions in Differential Games with Asynchronous Players' Horizons", paper presented at *the Thirteenth International Symposium on Dynamic Games and Applications,* Wroclaw, Poland, June 30 - 3 July, 2008.

"Environmental Problems in China: From Kyoto Protocol and Bali Roadmap to Foundations for a Cooperative Game-theoretic Solution" (with Y.X. Zhang), paper presented at *International Workshop on Modeling Technology Oriented RD&D Strategic Cooperation for Climate Change Mitigation: Methodological Issues and Alternative Policy Scenario*, Venice, Italy, 17-18 March 2008.

"Managing Catastrophe-bound Industrial Pollution with Game Theoretic Algorithm: The St Petersburg Initiative," Plenary Lecture presented at *First International Conference on Game Theory and Management,* St Petersburg, Russia, June 28-29, 2007 (Nobel Laureate Robert Aumann was another Plenary Speaker).

"A Dynamic Game-theoretic Solution to Catastrophe-bound Industrial Pollution Problems" Plenary Lecture presented at *The Second China Meeting on Game Theory and Applications*, Qingdao, China, 17-19 September, 2007.

"Cooperative Stochastic Differential Games" (with L. Petrosyan) Invited Lecture presented at the 3rd Pan Pacific Conference on Game Theory, Beijing, China, 20-22 October 2006.

"Dynamically Stable Cooperative Solutions in Differential Games with Randomly Furcating Payoffs" (with L. Petrosyan) paper presented at *the Twelve International Symposium on Dynamic Games and Applications*, Sophia Antipolis, France, 3-6 July, 2006.

"The Tenet of Transitory Compensation in Dynamically Stable Cooperation" (with L. Petrosyan) paper presented at the 13th IFAC Workshop on control and Applications of Optimization, Paris, 26-28 April 2006.

"Exact Solution to a Class of Stochastic Resource Extraction Problems" (with T. S. Cheng) presented at *International Scientific Annual Conference on Operations Research 2005*, Bremen, Germany, 7-9 September 2005.

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"Stochastic Pursuit Problems with Different Geometric Shapes of Reachable Sets," (with W. Yu), presented at the *First Workshop of the China Chapter of the International Society of Dynamic Games*, Qingdao, 8-9 August 2004.

"On Solutions to Cooperative Stochastic Differential Games," Plenary Lecture presented at the *First Workshop of the China Chapter of the International Society of Dynamic Games*, Qingdao, 8-9 August 2004.

"Subgame Consistent Solutions in Stochastic Games and Stochastic Differential Games," (with L. A. Petrosyan) presented at the *Second World Congress of the Game Theory Society*, Marseille, France, 5-9 July 2004.

"Recent Advances in Cooperative Differential Games," invited presentation at *GERAD Summer School on Differential Games*, Montreal, 14-18, June 2004.

"Subgame Consistent Dormant-firm Cartel," presented at *Third International Workshop on Agent-based Approaches in Economics and Social Complex Systems*, Kyoto, Japan, 27-29 May 2004.

"Endogenous Horizon Stochastic Randomly-furcating Differential Games," presented at Operations Research 2003 – *Annual International Conference of the German Operations Research Society*, Heidelberg, 3-5, September 2003.

"Infinite-horizon Stochastic Control for Problems with Randomly-furcating Payoffs," (with T.S. Cheng) presented at *Operations Research 2003 – Annual International Conference of the German Operations Research Society*, Heidelberg, 3-5, September 2003.

"Randomly Furcating Stochastic Differential Games: A Paradigm for Interactive Decision-making under Structure Uncertainty," Plenary Lecture of the (ICM) *International Congress of Mathematicians Satellite Conference on Games and Applications*, Qingdao, 14-17, August 2002. (Other Plenary Speakers included Nobel Laureates John Nash, Reinhard Selten and Robert Aumann).

"Infinite-time Stochastic Differential Games with Uncertain Number and Types of Players," presented at the *10th International Symposium on Dynamic Games and Applications*, St Petersburg, 8-11 July 2002.

"Randomly-Furcating Stochastic Differential Games under Structure-Uncertainty and Horizon Indefiniteness," Lecture presented at the *Honorary Doctorate Award Ceremony of St Petersburg University*, Russia, 11 June 2002.

"Infinite Overlapping-Generations Stochastic Differential Games with Uncertain Number and Types of Players," presented at the *5th International Conference on Optimization: Techniques and Applications*, Hong Kong, December, 2001.

"Proportional Time-Consistent Solutions in Differential Games," presented at the *International Conference on Logic, Game Theory and Social Choice*, St Petersburg, June 2001.

"Infinite Horizon Stochastic Differential Games with Uncertain Future Payoff Structures," presented at the 4th International Conference on Computational Intelligence and Multimedia Applications, Yokosuka City, 2001.

"A Renewable Resource Extraction Game with Uncertain Future Payoff Structures," presented at the *JSAI 2001 International Workshop on Agent-Based Approaches in Economics and Social Complex Systems*, Matsue, Japan, 2001.

"Dynamic Cooperative Games with Uncertain Payoffs," presented at the *Ninth International Symposium on Dynamic Game and Applications*, Adelaide, December 2000.

"Investors Behavior and Asset Price Dynamics," presented at the *Ninth International Symposium on Dynamic Game and Applications*, Adelaide, December 2000.

"Equilibrium Asset Dynamics with Holding-term Switching," presented at the *International Workshop on Decision & Control in Management Sciences in Honor of Professor Alain Haurie*, Montreal, October, 2000. (Invited)

"Differential Games with Uncertain Terminal Payoffs," presented at the *Seventh Viennese Workshop on Optimal Control, Dynamic Games and Nonlinear Dynamics*, Vienna, May 2000.

"A Multi-dimensional Stochastic Process for Generalized Asset Price Dynamics," presented at the *Seventh Viennese Workshop on Optimal Control, Dynamic Games and Nonlinear Dynamics*, Vienna, May 2000.

"Shapley Values for Games with Uncertain Terminal Payoffs," (with Leon Petrosjan), presented at The *First World Congress of the game Theory Society*, Bilbao, Spain, July, 2000.

"Differential Games with Uncertain Payoffs", (with Leon Petrosjan), presented at the *International Federation of Automatic Control Workshop on Control Applications of Optimization*, St Petersburg, July 2000.

"Dynamic Games with Random Duration and Uncertain Payoffs" (with Leon Petrosyan), presented at the 5th International Petrozavodsk Conference on Probabilistic Methods in Discrete Mathematics, 5-10, June, 2000.

"Concession Bargaining for Tansboundary Pollution Reduction," (with Steffen Jorgensen), presented at the *5th International Conference of the Decision Sciences Institute*, Athens, July 1999. (Invited)

"Strategic Concession", (with Steffen Jorgensen), presented at the *Far Eastern Meeting of the Econometric Society*, Singapore, July 1999.

"A Stochastic Differential Game of Institutional Investor Speculation," presented at the *Eighth International Symposium on Dynamic Games and Applications*, Maastricht, The Netherlands, July 1998.

"Institutional Investor Speculation: Theoretical Underpinnings and Applications in a Stochastic Differential Game Framework," presented at the *Fourth International Conference on Optimization: Techniques and Applications*, Perth, Australia, July 1998.

"A Strategic Concession Game," (with Steffen Jorgensen), presented at the *Conference on Microeconomics and Game Theory*, Copenhagen, Denmark, June 1997.

"Inter- and Intragenerational Resource Extraction and an Intergenerational Equity Rule," (with Steffen Jorgensen), presented at The *Sixth Viennese Workshop on Optimal Control, Dynamic Games, Nonlinear Dynamics and Adaptive Systems*, Vienna, Austria, May, 1997.

"A Class of Differential Games Which Admits a Feedback Solution with Linear Value Functions," presented at The *Sixth Viennese Workshop on Optimal Control, Dynamic Games, Nonlinear Dynamics and Adaptive Systems*, Vienna, Austria, May, 1997.

"Transboundary Pollution Management: A Concession Equilibrium," (with Steffen Jorgensen), presented at The *Optimization Days*, Montreal, Canada, May 1997. (Invited)

"Inter- and intra-generational Resource Extraction," (with Steffen Jorgensen), presented at the *Seventh International Symposium on Dynamic Games and Applications*, Kanagawa, Japan, December, 1996.

"Institutional Evolution and Emerging Markets: A Model of Stock Price Dynamics in China," (with Jessie Poon), presented at the *Symposium on The Expansion Method*, Odense, Denmark, October, 1996.

"Price Dynamics in Emerging Stock Markets," presented at the *20th Symposium on Operations Research*, Passau, Germany, Sept., 1995.

"A Theory of the Price Dynamics of China's Stock Markets," presented at the *2nd Conference on the Development of China's Securities Markets*, Shanghai, China, Dec. 1994.

"Derivation of Feedback Solutions for a General Class of Infinite Horizon Games Circumventing the Hamilton-Jacobi-Bellman Equations," (with L.Y. Yeung), presented at the *International Conference on Operations Research*, Berlin, Germany Aug./Sept. 1994.

"Common-Pool Resource Extraction: Dynamic Equilibria," presented at the *International Conference on Operations Research*, Berlin, Germany Aug./Sept. 1994.

"Common-Pool Resource Extraction: Feedback Nash Equilibrium Solution," presented at the *Sixth International Symposium on differential Games and Applications*, St. Jovite, Quebec, Canada, July 1994. (Invited)

"Price Dynamics in China's Emerging Stock Markets," presented at the *Western Economic Association International Pacific Rim Conference*, Hong Kong, Jan. 1994.

"On Degenerate and Non-degenerate Correspondences of Differential Games with a Feedback Nash Equilibrium," presented at the *32nd IEEE Conference on Decision and Control*, San Antonio, Texas, Dec., 1993.

"Reconstruction of Public Finance in a Stochastic Differential Game Framework," presented at *the 18th Symposium on Operations Research*, Koeln, Germany, Sept. 1993.

"The Shenzhen Stock Market: Current State and a Forecasting Model," (with Michael Cheung and Lai-Yee Yeung), presented at the *14th International Symposium on Asian Studies*, Hong Kong, July, 1992.

"Capital Accumulation Subject to Pollution Control: A Differential Game with a Feedback Nash Equilibrium," (with Michael Cheung), presented at the *Fifth International Symposium on Differential Games and Applications*, Geneva, Switzerland, July, 1992.

"A Stochastic Model of Foreign Exchange Dynamics and An Exact Option Pricing Formula," (with Michael Cheung), presented at the *17th Symposium on Operations Research*, Hamburg, Germany, August, 1992.

"On the Use of Geometric Brownian Motion in Financial Analysis," (with Michael Cheung and Alfred Lai), presented at the *17th Symposium on Operations Research*, Hamburg, Germany, August, 1992.

"Characterizing the Solution of an Average Cost Minimization Problem with returns to scale and a Decomposition Technique," (with Michael Cheung), presented at the *Sixteenth Symposium on Operations Research*, Trier, Germany, September 1991.

"Industrial Pollution Management in a Differential Game Framework," presented at the *International Conference on Operations Research*, Vienna, August, 1990.

"A Game Theoretical Approach to Public Investment Versus Private Investment," presented at the *Fourth International Symposium on Differential Games and Applications*, Helsinki, August, 1990. (Invited)

"Relation Wages and Inflation Revisited (with Hafiz Akhand), Presented at the *Eighteenth Atlantic Canadian Economic Association Conference*, Saint John, 1989.

"Monetary Rules as Optimal Incentive Strategies," (invited) presented at the *6th International Federation of Automatic Control Symposium on Dynamic Modelling and Control of National Economics, Edinburgh*, June, 1989.

"Nonmalleable Capital and Efficient Allocation of Nonrenewable Resource," (with John Rowse), presented at the *Joint National Meeting of the Institute of Management Science, Canadian Operational Research Society and ORSA*, Vancouver, May, 1989.

"Knowledge Based Product Innovation: A Closed-Loop Feedback Nash Equilibrium Approach," presented at the *Joint International Conference of the Association of European Operational Research Societies and the Institute of Management Sciences*, Paris, July, 1988.

"Economic Exploitation of Non-renewable Resources in Random Environments - A Stochastic Hotelling Rule," (with Charles Plourde), presented at the *Twentieth International Atlantic Economic Conference*, Washington, D.C., August, 1985.

"Predator-Prey Models in Fishery Economics" (with Charles Plourde) (invited) presented at *Colloque d'économie des resources naturalles*, Laval, Quebec, June 1984.

"A Stochastic Model of Industrial Pollution Management," (with Charles Plourde), presented at the Eighteenth International Atlantic Economic Conference, Montreal, October, 1984.

HIGHLIGHTS OF SCIENTIFIC CONTRIBUTIONS

GROUND-BREAKING CONTRIBUTION

(1) Cooperative stochastic dynamic games and Cooperative Subgame Consistency

The work of Yeung and Petrosyan [D. Yeung and L. Petrosyan: Subgame Consistent Cooperative Solution in Stochastic Differential Games, *Journal of Optimization Theory and*

Applications 2004] represents a breakthrough in the field of cooperative stochastic differential games. In particular, a generalized theorem for the derivation of analytically tractable "subgame consistent" solutions is presented. This contribution has made possible the rigorous study of dynamic stochastic cooperation.

This work lead to the world's first ever book devoted to cooperative stochastic differential games. [D. Yeung and L. Petrosyan: *Cooperative Stochastic Differential Games*, Springer 2006].

Praises for the book from two eminent dynamic game theorists stated that:

> " '*Cooperative Stochastic Differential Games' by Profs. Yeung and Petrosyan, two of the world's leading experts and pioneers in the field of dynamic games, promises to be the first substantial and basic treatment of cooperative games in a stochastic setting. This extremely difficult topic is addressed in novel and rigorous ways, developing some of the authors' earlier ideas in a comprehensive manner which will make this book the 'bible' of the field for years to come*." (George Leitmann, Professor of the Graduate School, University of California at Berkeley)

> "*This potentially classic text is written by the world's best cooperative dynamic game theorist and its top cooperative stochastic differential game theorist. A general theorem (of the authors') deriving analytically tractable 'payoff distribution procedures' for subgame consistent solutions represents a breakthrough in the rigorous study of cooperative stochastic differential games. The book's advancements in dynamic stability – especially in subgame consistency – mark a continuation of the great tradition of A. Lyapunov, L. S. Pontryagin and V. Zubov*." (Vladimir Mazalov, Director of the Institute of Applied Mathematical Research, Russian Academy of Sciences).

This volume adds to the analysis of cooperative games by extending it to a stochastic dynamic framework and providing a generalized theorem for obtaining dynamically stable solutions. The introduction of the concept of "Cooperative Subgame Consistency" allows the derivation of cooperative solutions which extension to a subgame at a later starting time and a state brought about by prior optimal behaviors would remain optimal. In particular, it implies that the specific optimality principle agreed upon at the outset must remain effective at any instant of time throughout the game along the optimal state trajectory.

The book "Subgame Consistent Cooperation: A Comprehensive Treatise, *Springer*, 2016" is the world's first book on Subgame Consistent Cooperation, which is a comprehensive treatise on the concepts and solution mechanisms of their pioneering work on cooperative subgame consistency. In the book's back-cover praise Vladimir Mazalov stated, "*The 2004 Nobel Economics Prize was given to works in economic policies under the concept of time consistency with mathematical construction less general, rigorous and precise than that later developed in this book. .... In terms of advancement in practical applications this book is highly important theoretically and technically on top of economic interpretation*."

(Vladimir Mazalov, Director of the Institute of Applied Mathematical Research, Russian Academy of Sciences).

Subgame consistent solutions for a list of new game paradigms developed by D. Yeung and L. Psetrosyan are given in item (3) below.

GROUND-BREAKING CONTRIBUTION

(2) A new field in dyanmic games
Durable-strategies Dynamic Games

Durable strategies that have prolonged effects are prevalent in real-world situations. Revenue-generating investments, toxic waste disposal, long-lived goods, regulatory measures, coalition agreements, diffusion of knowledge, advertisement and investments to accumulate physical capital are concrete examples of durable strategies. In reality, the existence of durable strategies in decision-making schemes is less the exception and more the rule.

The paradigm of 'durable-strategies dynamic games' provides a new perspective approach with a new class of strategies in dynamic game theory. The addition of durable strategies in the strategies set of dynamic games does not only make game theory applications possible for analysis of real-life problems with durable strategies but also establish a new theoretical framework with novel optimization theory, solution concepts and mathematical game techniques. This paradigm fills a significant gap in conventional dynamic game theory. The establishment of durable-strategies dynamic games provides an augmentation of dynamic game theory with durable strategies in decision-making schemes.

A comprehensive collection of the new game paradigm of durable-strategies dynamic games developed were given in:
D. Yeung and L. Petrosyan: Cooperative Dynamic Games with Control Lags, Dynamic Games and Applications, 2019.
L. Petrosyan and D. Yeung: Cooperative Dynamic Games with Durable Controls: Theory and Application, Dynamic Games and Applications, 2020.
D. Yeung and L. Petrosyan: Asynchronous Horizons Durable-Strategies Dynamic Games and Tragedy of Cross-Generational Environmental Commons, International Game Theory Review, 2021.
D. Yeung and L. Petrosyan: Generalized dynamic games with durable strategies under uncertain planning horizon, Journal of Computational and Applied Mathematics, 2021.

The book "Durable-strategies Dynamic Games: Theory, Solution Techniques and Applications, Springer 2022" by D. Yeung and L. Petrosyan is the world's first text about the new paradigm of durable-strategies dtnamic games.

Praises for the book from eminent dynamic game theorists stated that:
"*Not too many new concepts have been introduced in dynamic games since their inception. The introduction of the concept of durable strategies changes this trend and yields important contributions to environmental and business applications*."
(Dušan M Stipanović, Professor, University of Illinois at Urbana-Champaign)

"*Before this book, the field simply did not realize that most of our strategies are durable and entail profound effects in the future. Putting them into the mathematical framework of dynamic games is a great innovative effort*."
(Vladimir Turetsky, Professor, Ort Braude College)
"*Durable-strategies Dynamic Games is truly a world-leading addition to the field of dynamic games. It is a much needed publication to tackle increasingly crucial problems under the reality of durable strategies*."
(Vladimir Mazalov, Director of Mathematical Research, Russian Academy of Sciences & President of the International Society of Dynamic Games)

(3) The establishment of subgame consistent solutions for the following games developed by D. Yeung and L. Petrosyan

(i) Subgame-consistent Solution for Random-horizon Cooperative Dynamic Games
D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. Journal of Optimization Theory and Applications, Vol. 150, pp78-97, 2011.

(ii) Subgame-consistent Solution Cooperative Stochastic Dynamic Games under Uncertainty in Payoff Structures
D.W.K. Yeung and L. A. Petrosyan: Subgame-consistent Cooperative Solutions in Randomly Furcating Stochastic Dynamic Games. Mathematical and Computer Modelling, Vol 57, pp.976–991, 2013.

(iii) Subgame consistent discrete-time stochastic dynamic games
D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Solutions for Cooperative Stochastic Dynamic Games. Journal of Optimization Theory and Applications, Vol. 145, 2010, pp. 579-596.

(iv) Subgame-consistent Solution for Random-horizon Cooperative Stochastic Dynamic Games under Uncertainty in Payoff Structures
D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solutions For Randomly Furcating Stochastic Dynamic Games With Uncertain Horizon. International Game Theory Review, Vol. 16, pp.1440012.01-1440012.29, 2014.

(v) Subgame-consistent Solution for Dynamic Cooperation under Non-transferrable Payoffs
D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solution for NTU Dynamic Games via Variable Weights, Automatica, 2015.

(4) New paradigms in game theory –

(i) Random Horizon Dynamic Games
This analysis extended discrete-time dynamic cooperative games to a random horizon framework. In many game situations, the terminal time of the game is not known with certainty. Examples of this kind of problems include uncertainty in the renewal of lease, the terms of offices of elected authorities/directorships, contract renewal and continuation of agreements subjected to periodic negotiations. A dynamic programming technique for solving inter-temporal problems with random horizon is developed to serve as the foundation of solving the game problem. To characterize a noncooperative equilibrium, a set of random duration discrete-time Hamilton-Jacobi-Bellman equations is derived.

D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. *Journal of Optimization Theory and Applications*, Vol. 150, pp78-97, 2011.

(ii) Randomly Furcating Stochastic Differential Games
Through the addition of stochastic elements via branching payoffs a fruitful approach to game modeling under uncertainty is offered. In particular, novel mathematical theorems yielding time-varying controls in infinite horizon problems are obtained. This work extended the classic contributions of Bellman and Isaacs (in the 1950s) and the stochastic analysis of Fleming and Rishel (in the 1970s) to stochastic differential games with randomly branching payoffs.

D. Yeung: Infinite Horizon Stochastic Differential Games with Branching Payoffs, *Journal of Optimization Theory and Applications* 2001.
D. Yeung: Randomly Furcating Stochastic Differential Games, in L. Petrosyan and D.W.K. Yeung (eds.): *ICM Millennium Lectures on Games*, Springer-Verlag, Berlin, 2003.

(4) Novel Contributions in Control Theory

(i) Random Horizon Bellman's Equation

D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. Journal of Optimization Theory and Applications, Vol. 150, pp78-97, 2011.

(ii) Random Horizon Stochastic Bellman Equation under Uncertain Payoff Structures

D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solutions For Randomly Furcating Stochastic Dynamic Games With Uncertain Horizon. International Game Theory Review, Vol. 16, 2014, pp.1440012.01-1440012.29.

(iii) Discrete-time Bellman Equations for Optimization under Durable-strategies

L. Petrosyan and D. Yeung: Cooperative Dynamic Games with Durable Controls: Theory and Application, Dynamic Games and Applications, 2020.

(5) Dynamic Consumer Theory
Inter-temporal Roy's Identities and Stochastic Dynamic Slutsky Equations

The development of the classic Slutsky's analysis on consumer theory to a stochastic dynamic framework provides novel results and a new venue of research with much higher realism and relevance for analysis in consumer theory. It extends the conventional static optimal consumption theories to a dynamic framework with the possibility of accommodating various combinations of three major types of uncertainty in consumption behavior – stochastic income, random life-span and uncertain preferences. These extensions inject some crucial, realistic and intrinsic characteristics of the consumer decision into the analysis of consumer theory. A series of inter-temporal Roy's identities and stochastic dynamic Slutsky equations are formulated. These include --- *Inter-temporal Roy's Identity, Inter-temporal Roy's Identity under Stochastic Income, Inter-temporal Roy's Identity under Stochastic Life-span, Inter-temporal Roy's Identity under Stochastic Income and Life-span, Inter-temporal Roy's Identity under Stochastic Preferences, Inter-temporal Roy's Identity under Stochastic Life-span and Preferences, Inter-temporal Roy's Identity under Stochastic Income and Preferences, Inter-temporal Roy's Identity under Stochastic Income, Life-span and Preferences, Dynamic Slutsky Equation, Dynamic Slutsky Equation under Stochastic Income, Dynamic Slutsky Equation under Stochastic Life-span, Dynamic Slutsky Equation under Stochastic Income and Life-span, Dynamic Slutsky Equation under Stochastic Preferences, Dynamic Slutsky Equation under Stochastic Life-span and Preferences, Dynamic Slutsky Equation under Stochastic Income and Preferences, and Dynamic Slutsky Equation under Stochastic Income, Life-span and Preferences.*

This work lead to the world's first ever book devoted on Dynamic Consumer Theory. [D.W.K. Yeung*: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations*, Nova Science Publishers, New York, 2015]. Acclaim for the book from two eminent applied mathematicians stated that:

> *"This book is an innovative addition to the study of consumer theory. It expands the conventional static consumer theory into a stochastic dynamic analysis – a move that*

makes the study of optimal consumption decision much more realistic." (Leon Petrosyan, St Petersburg State University, Dean of Faculty of Applied Mathematics)

"The stochastic dynamic Slutsky equations in this book formed a new milestone in consumer theory after the classic work of Slutsky. ..... it is highly apt to call the latter 'Slutsky-Yeung equations'". (Vladimir Mazalov, Director of Karelian Institute of Applied mathematical Research, Russian Academy of Sciences)

D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.
D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141.
D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5, 2014, pp.263-284.
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

(7) World's First Texts

(i) The world's first text on cooperative stochastic differential games:
D. Yeung and L. Petrosyan: Cooperative Stochastic Differential Games, Springer 2006.

(ii) The world's first text on subgame consistent economic optimization:
D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Economic Optimization: An Advanced Cooperative Dynamic Game Analysis, Boston: Birkhäuser 2012.

(iii) The world's first text on dynamic consumer theory:
D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.

(iv) The world's first text on subgame consistent cooperation:
D.W.K.Yeung and L.A. Petrosyan.: Subgame Consistent Cooperation – A Comprehensive Treatise, Springer, 2016.

(v) The world's first text on dynamic Shapley value and dynamic Nash bargaining:
D.W.K. Yeung and L.A. Petrosyan: Dynamic Shapley Value and Dynamic Nash Bargaining, Nova Science, New York, 2018.

(vi) The world's first text on durable-strategies dynamic games
D.W.K. Yeung and L.A. Petrosyan: Durable-strategies Dynamic Games: Theory, Solution Techniques and Applications, Springer, 2021.

LIST OF MATHEMATICAL FORMULAE DEVELOPED

Part A: Control Theory

Theorem A1: Random-horizon Bellman Equation
Theorem A2: Random-horizon Stochastic Bellman Equation under Uncertain Future Payoff Structures
Theorem A3: Dynamic Programming with Control Lags

Theorem A4: Dynamic Optimization Theorem Under Random Horizon and Durable Strategies

Part B: Game Theory

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Theorem B3: Nontransferable Individual Payoff in Stochastic Dynamic Cooperation
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Theorem B5: Subgame-consistent Payoff Distribution Procedure (PDP) for Discrete-time Stochastic Dynamic Cooperation
Theorem B6: Subgame-consistent PDP for Continuous-time Stochastic Dynamic Cooperation
Theorem B7: Subgame-consistent PDP for Random-horizon Dynamic Cooperation
Theorem B8: Subgame-consistent PDP for Discrete-time Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures
Theorem B9: Subgame-consistent PDP for Continuous-time Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures
Theorem B10: Subgame-consistent PDP for Random-horizon Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures
Theorem B11: Subgame-consistent Solution Mechanism for Dynamic Cooperation under Non-transferrable (NTU) Payoffs
Theorem B12: Subgame-consistent Solution Mechanism for Stochastic Dynamic Cooperation under Non-transferrable (NTU) Payoffs
Theorem B13: Hamilton-Jacobi-Bellman Equations for Dynamic Games with Durable Controls
Theorem B14: Subgame-consistent PDP for Cooperative Dynamic Games with Durable Controls
Theorem B15. Hamilton-Jacobi-Bellman Equations for Dynamic Games under Random Horizon and Durable Controls
Theorem B16. Subgame-consistent PDP for Cooperative Dynamic Games under Random Horizon and Durable Controls

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C2. Inter-temporal Roy’s Identity under Stochastic Life-span
C3. Inter-temporal Roy’s Identity under Stochastic Income
C4. Inter-temporal Roy’s Identity under Stochastic Income and Life-span
C5. Inter-temporal Roy’s Identity under Stochastic Preferences
C6. Inter-temporal Roy’s Identity under Stochastic Life-span and Preferences
C7. Inter-temporal Roy’s Identity under Stochastic Income and Preferences
C8. Inter-temporal Roy’s Identity under Stochastic Income, Life-span and Preferences
C9. Dynamic Slutsky Equation
C10. Dynamic Slutsky Equation under Stochastic Life-span
C11. Dynamic Slutsky Equation under Stochastic Income
C12. Dynamic Slutsky Equation under Stochastic Income and Life-span
C13. Dynamic Slutsky Equation under Stochastic Preferences
C14. Dynamic Slutsky Equation under Stochastic Life-span and Preferences
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Part D. Number Theory

D1. The Number of Embedded Coalitions

D2. The Number of Embedded Coalitions where the Position of the Individual Player Counts

Part E. Probability Density Function

E1. Stationary Probability Density Function of the Lotka-Volterra-Yeung Stochastic Food-chain

E2. Stationary Probability Density Function of Generalized Stochastic Food-chain of the Lotka-Volterra-Yeung Type

ADDENDUM TO
Curriculum Vitae of David W. K. Yeung

Mathematical Formulae Developed

One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers... ~Heinrich Hertz (1847-1894)

Part A: Control Theory

Theorem A1. (Random-horizon Bellman Equation)

$$V(T+1,x)=q_{T+1}(x),$$

$$V(T,x)=\max_{u_T}\left\{\quad g_T(x,u_T)+V[T+1,f_T(x,u_T)]\quad\right\},$$

$$V(\tau,x)=\max_{u_\tau}\left\{\quad g_\tau(x,u_\tau)+\frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta}q_{\tau+1}[f_\tau(x,u_\tau)]\right.$$

$$\left.+\frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta}V[\tau+1,f_\tau(x,u_\tau)]\quad\right\},\ \text{for } \tau\in\{1,2,\cdots,T-1\}. \qquad \blacksquare$$

Origin and Application

The value function $V(k,x)$ denote the maximal value of the expected payoffs for the problem of maximizing

$$E\left\{\quad\sum_{k=\tau}^{\hat{T}}\quad g_k(x_k,u_k)\delta_1^k+q_{\hat{T}+1}(x_{\hat{T}+1})\delta_1^{\hat{T}+1}\quad\right\}$$

$$=\sum_{\hat{T}=\tau}^{T}\frac{\theta_{\hat{T}}}{\sum_{\zeta=\tau}^{T}\theta_\zeta}\left\{\quad\sum_{k=\tau}^{\hat{T}}\quad g_k(x_k,u_k)\delta_1^k+q_{\hat{T}+1}(x_{\hat{T}+1})\delta_1^{\hat{T}+1}\quad\right\}$$

subject to

$$x_{k+1}=f_k(x_k,u_k),\qquad x_k=x,$$

where state $x_\tau=x$, $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\theta_1,\theta_2,\cdots,\theta_T\}$.

Reference: *D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. Journal of Optimization Theory and Applications, Vol. 150, pp78-97, 2011.*

Theorem A2 (Random-horizon Stochastic Bellman Equation under Uncertain Future Payoff Structures)

$V^{(\sigma_{T+1})}(T+1,x) = q_{T+1}(x)$,

$$V^{(\sigma_T)}(T,x) = \max_{u_T} E_{\vartheta_T}\left\{ \quad g_T\left(x,u_T;\theta_T^{\sigma_T}\right) + V^{(\sigma_{T+1})}[T+1, f_T(x,u_T)+\vartheta_T] \quad \right\}$$

$$= E_{\vartheta_T}\left\{ \quad g_T\left(x,\varphi_T^{(\sigma_T)*}(x);\theta_T^{\sigma_T}\right) + V^{(\sigma_{T+1})}\left[T+1, f_T\left(x,\varphi_T^{(\sigma_T)*}(x)\right)+\vartheta_T\right] \quad \right\},$$

$$V^{(\sigma_\tau)}(\tau,x) = \max_{u_\tau} E_{\vartheta_\tau}\left\{ \quad g_\tau\left(x,u_\tau;\theta_\tau^{\sigma_\tau}\right) + \frac{\varpi_\tau}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} q_{\tau+1}[f_\tau(x,u_\tau)+\vartheta_\tau]\right.$$

$$\left. + \frac{\sum_{\zeta=\tau+1}^{T}\varpi_\zeta}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} \sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}} \lambda_{\tau+1}^{\sigma_{\tau+1}} V^{(\sigma_{\tau+1})}[\tau+1, f_\tau(x,u_\tau)+\vartheta_\tau] \quad \right\}$$

$$= E_{\vartheta_\tau}\left\{ \quad g_\tau\left(x,\varphi_\tau^{(\sigma_\tau)*}(x);\theta_\tau^{\sigma_\tau}\right) + \frac{\varpi_\tau}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} q_{\tau+1}\left[f_\tau\left(x,\varphi_\tau^{(\sigma_\tau)*}(x)\right)+\vartheta_\tau\right]\right.$$

$$\left. + \frac{\sum_{\zeta=\tau+1}^{T}\varpi_\zeta}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} \sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}} \lambda_{\tau+1}^{\sigma_{\tau+1}} V^{(\sigma_{\tau+1})}\left[\tau+1, f_\tau\left(x,\varphi_\tau^{(\sigma_\tau)*}(x)\right)+\vartheta_\tau\right] \quad \right\},$$

for $\tau \in \{1,2,\cdots,T-1\}$. ■

Origin and Application

The value function $V^{(\sigma_\tau)}(\tau,x)$ denote the maximal value of the expected payoffs for the problem of maximizing

$$E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\left\{ \quad \sum_{\hat{T}=1}^{T}\varpi_{\hat{T}}\left[\quad \sum_{k=1}^{\hat{T}} \quad g_k^i[x_k,u_k;\theta_k] + q(x_{\hat{T}+1}) \quad \right] \right\},$$

subject to $x_{k+1} = f_k(x_k,u_k)+\vartheta_k$,
where $E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}$ is the expectation operation with respect to the random variables $\theta_1,\theta_2,\cdots,\theta_T$ and $\vartheta_1,\vartheta_2,\cdots,\vartheta_T$.

Reference: *D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solutions For Randomly Furcating Stochastic Dynamic Games With Uncertain Horizon. International Game Theory Review, Vol. 16, 2014, pp.1440012.01-1440012.29.*

Theorem A3. Dynamic Optimization Technique for Durable Controls

$V\big(T+1,x;\bar{u}_{(T+1)-}\big) = q_{T+1}\big(x_{T+1};\bar{u}_{(T+1)-}\big)\delta_1^{T+1}$,

$$V(k,x;\bar{u}_{k-}) = \max_{u_k,\bar{u}_k}\left\{ \quad g_k(x,u_k,\bar{u}_k;\bar{u}_{k-})\delta_1^k + V\big[k+1, f_k(x,u_k,\bar{u}_k;\bar{u}_{k-});\bar{u}_{(k+1)-}\big] \quad \right\}$$

$$= \max_{u_k,\bar{u}_k}\left\{ \quad g_k(x,u_k,\bar{u}_k;\bar{u}_{k-})\delta_1^k + V\big[k+1, f_k(x,u_k,\bar{u}_k;\bar{u}_{k-});\bar{u}_k,\bar{u}_{(k+1)-}\cap\bar{u}_{k-}\big] \quad \right\},$$

for $k \in \{1,2,\cdots,T\}$. ■

Origin and Application

The function $V(k,x;\bar{u}_{k-})$ represents the maximal value of the payoff for the problem of maximizing

$$\sum_{t=k}^{T} g_t(x_t,u_t,\bar{u}_k;\bar{u}_{t-})\,\delta_1^t + q_{T+1}\big(x_{T+1};\bar{u}_{(T+1)-}\big)\delta_1^{T+1}$$

subject to

$$x_{t+1} = f_t(x_t, u_t, \bar{u}_t; \bar{u}_{t-}),$$

with state $x_k = x$ and previously executed durable controls $\bar{u}_{k-}$.

References**:** *D.W.K. Yeung, L.A. Petrosyan (2020): Cooperative Dynamic Games with Durable Controls: Theory and Application, Dynamic Games and Applications, Doi:10.1007/s13235-019-00336-w. 9(2), 550-567, 2019, https://doi.org/10.1007/s13235-018-0266-6.*

D.W.K. Yeung, L.A. Petrosyan (2019): Cooperative Dynamic Games with Control Lags, Dynamic Games and Applications, 9(2), 550-567, https://doi.org/10.1007/s13235-018-0266-6.
D.W.K. Yeung, L.A. Petrosyan (2022): Durable-strategies Dynamic Games: Theory, Solution Techniques and Applications, Springer, Switzerland, 303pp, 2022, ISBN 9783030927417.

Theorem A4. Dynamic Optimization Theorem Under Random Horizon and Durable Strategies

$$V\big(T+1, x; \bar{u}_{(T+1)-}\big) = q_{T+1}\big(x; \bar{u}_{(T+1)-}\big)\delta_1^{T+1},$$

$$V(T, x; \bar{u}_{T-}) = \max_{u_T, \bar{u}_T}\Big\{ \quad g_T(x, u_T, \bar{u}_T; \bar{u}_{T-})\delta_1^T$$
$$+q_{T+1}\big[f_T(x, u_T, \bar{u}_T; \bar{u}_{T-}); \bar{u}_T, \bar{u}_{(T+1)-} \cap \bar{u}_{T-}\big]\delta_1^{T+1} \quad \Big\},$$

$$V(\tau, x; \bar{u}_{\tau-}) = \max_{u_\tau, \bar{u}_\tau}\Big\{ \quad g_\tau(x, u_\tau, \bar{u}_\tau; \bar{u}_{\tau-})\delta_1^\tau$$
$$+\frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta} q_{\tau+1}\big[f_T(x, u_\tau, \bar{u}_\tau; \bar{u}_{\tau-}); \bar{u}_\tau, \bar{u}_{(\tau+1)-} \cap \bar{u}_{\tau-}\big]\delta_1^{\tau+1}$$
$$+\frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta} V\big(\tau+1, f_\tau(x, u_\tau, \bar{u}_\tau; \bar{u}_{\tau-}); \bar{u}_\tau, \bar{u}_{(\tau+1)-} \cap \bar{u}_{\tau-}\big) \quad \Big\},$$ for $\tau \in \{1,2,\cdots,T-1\}$. ∎

Origin and Application

The value function $V(\tau, x; \bar{u}_{\tau-})$ denote the maximal value of the expected payoffs for the problem of maximizing

$$E\Big\{ \quad \sum_{k=\tau}^{\hat{T}} \quad g_k(x_k, u_k, \bar{u}_k; \bar{u}_{k-})\delta_1^k + q_{\hat{T}+1}\big(x_{\hat{T}+1}; \bar{u}_{(\hat{T}+1)-}\big)\delta_1^{\hat{T}+1} \quad \Big\}$$
$$= \sum_{\hat{T}=\tau}^{T}\frac{\theta_{\hat{T}}}{\sum_{\zeta=\tau}^{T}\theta_\zeta}\Big\{ \quad \sum_{k=\tau}^{\hat{T}} \quad g_k(x_k, u_k, \bar{u}_k; \bar{u}_{k-})\delta_1^k + q_{\hat{T}+1}\big(x_{\hat{T}+1}; \bar{u}_{(\hat{T}+1)-}\big)\delta_1^{\hat{T}+1} \quad \Big\}$$

subject to

$x_{k+1} = f_k(x_k, u_k, \bar{u}_k; \bar{u}_{k-}), \qquad x_k = x,$

where $q_{\hat{T}+1}\big(x_{\hat{T}+1}; \bar{u}_{(\hat{T}+1)-}\big)$ is the terminal payment in stage $\hat{T}+1$,

state $x_\tau = x$ and previously executed controls $u_{\tau-}$,

$\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\theta_1, \theta_2, \cdots, \theta_T\}$.

References**:** *D.W.K. and Petrosyan, L.A.: Generalized dynamic games with durable strategies under uncertain planning horizon, Journal of Computational and Applied Mathematics, 395 (2021) 113595.*

Part B: Game Theory

Theorem B1. (Random-horizon (Hamilton-Jacobi-Bellman) HJB Equations)

$$V^i(T,x)=\max_{u_T^i}\Big\{\quad g_T^i\big[x,\varphi_T^1(x),\varphi_T^2(x),\cdots,\varphi_T^{i-1}(x),u_T^i,\varphi_T^{i+1}(x),\cdots,\varphi_T^n(x)\big]$$

$$+q_{T+1}^i\Big[f_k\Big(x,\varphi_k^1(x),\varphi_k^2(x),\cdots,\varphi_k^{i-1}(x),u_k^i,\varphi_k^{i+1}(x),\cdots,\varphi_k^n(x)\Big)\Big]\quad\Big\},$$

$$V^i(\tau,x)=\max_{u_\tau^i}\Big\{\quad g_\tau^i[x,\varphi_\tau^1(x),\varphi_\tau^2(x),\cdots,\varphi_\tau^{i-1}(x),u_\tau^i,\varphi_\tau^{i+1}(x),\cdots,\varphi_\tau^n(x)]$$

$$+\frac{\theta_\tau}{\sum_{\zeta=\tau}^T\theta_\zeta}q_{\tau+1}^i\Big[f_k\Big(x,\varphi_k^1(x),\varphi_k^2(x),\cdots,\varphi_k^{i-1}(x),u_k^i,\varphi_k^{i+1}(x),\cdots,\varphi_k^n(x)\Big)\Big]$$

$$+\frac{\sum_{\zeta=\tau+1}^T\theta_\zeta}{\sum_{\zeta=\tau}^T\theta_\zeta}V^i\Big[\tau+1,f_k\Big(x,\varphi_k^1(x),\varphi_k^2(x),\cdots,\varphi_k^{i-1}(x),u_k^i,\varphi_k^{i+1}(x),\cdots,\varphi_k^n(x)\Big)\Big]\quad\Big\},\quad(3.1)$$

for $\tau\in\{1,2,\cdots,T-1\}$. ■

Origin and Application

The value function $V^i(\tau,x)$ denote the Feedback Nash equilibrium payoff of player $i\in N$ for the game problem

$$\max_{u^i}E\Big\{\quad\sum_{k=1}^{\hat{T}}\quad g_k^i[x_k,u_k^1,u_k^2,\cdots,u_k^n]+q_{\hat{T}+1}^i(x_{\hat{T}+1})\quad\Big\}$$

$$=\sum_{\hat{T}=1}^T\theta_{\hat{T}}\Big\{\quad\sum_{k=1}^{\hat{T}}\quad g_k^i[x_k,u_k^1,u_k^2,\cdots,u_k^n]+q_{\hat{T}+1}^i(x_{\hat{T}+1})\quad\Big\},\quad i\in N,$$

subject to $x_{k+1}=f_k(x_k,u_k^1,u_k^2,\cdots,u_k^n)$.

Reference*: *D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. Journal of Optimization Theory and Applications, Vol. 150, pp78-97, 2011.

Theorem B2. (Random-horizon Stochastic HJB Equations under Uncertain Future Payoff Structures)

$$V^{(\sigma_{T+1})i}(T+1,x)=q_{T+1}^i(x),$$

$$V^{(\sigma_T)i}(T,x)=\max_{u_T^i}E_{\vartheta_T}\Big\{\quad g_T^i[x,\varphi_T^{(\sigma_T)1*}(x),\varphi_T^{(\sigma_T)2*}(x),\cdots,\varphi_T^{(\sigma_T)i-1*}(x),u_T^{(\sigma_T)i},\varphi_T^{(\sigma_T)i+1*}(x),\cdots$$

$$\cdots,\varphi_T^{(\sigma_T)n*}(x);\theta_T^{\sigma_T}]+V^{(\sigma_{T+1})i}[T+1,f_T(x,\underline{\varphi}_T^{(\sigma_T)*\neq i})+\vartheta_T]\quad\Big\},$$

$$V^{(\sigma_\tau)i}(\tau,x)=\max_{u_\tau^i}E_{\vartheta_\tau}\Big\{\quad g_\tau^i[x,\varphi_\tau^{(\sigma_\tau)1*}(x),\varphi_\tau^{(\sigma_\tau)2*}(x),\cdots,\varphi_\tau^{(\sigma_\tau)i-1*}(x),u_\tau^{(\sigma_\tau)i},\varphi_\tau^{(\sigma_\tau)i+1*}(x),\cdots$$

$$\cdots,\varphi_\tau^{(\sigma_\tau)n*}(x);\theta_\tau^{\sigma_\tau}]+\frac{\varpi_\tau}{\sum_{\zeta=\tau}^T\varpi_\zeta}q_{\tau+1}[f_\tau(x,\underline{\varphi}_\tau^{(\sigma_\tau)*\neq i}(x))+\vartheta_\tau]$$

$$+\frac{\sum_{\zeta=\tau+1}^T\varpi_\zeta}{\sum_{\zeta=\tau}^T\varpi_\zeta}\sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}}\lambda_{\tau+1}^{\sigma_{\tau+1}}V^{(\sigma_{\tau+1})}\Big[\tau+1,f_\tau\Big(x,\underline{\varphi}_\tau^{(\sigma_\tau)*\neq i}(x)\Big)+\vartheta_\tau\Big]\quad\Big\},\ \tau\in\{1,2,\cdots,T-1\};$$

for $\sigma_t\in\{1,2,\cdots,\eta_t\}$, $t\in\{1,2,\cdots,T\}$ and $i\in N$;

where $\underline{\varphi}_t^{(\sigma_t)*\neq i}(x)$

$$=\Big[\varphi_t^{(\sigma_t)1*}(x),\varphi_t^{(\sigma_t)2*}(x),\cdots,\varphi_t^{(\sigma_t)i-1*}(x),u_t^{(\sigma_t)i},\varphi_t^{(\sigma_t)i+1*}(x),\cdots,\varphi_t^{(\sigma_t)n*}(x)\Big];$$

for $t\in\{1,2,\cdots,T\}$. ■

Origin and Application

The value function $V^{(\sigma_\tau)i}(\tau,x)$ denote the Feedback Nash equilibrium payoff of player $i \in N$ for the game problem

$$\max_{u^i} E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\left\{ \sum_{\hat{T}=1}^{T} \varpi_{\hat{T}} \left[\sum_{k=1}^{\hat{T}} g_k^i[x_k,u_k^1,u_k^2,\cdots,u_k^n;\theta_k] + q^i\ (x_{\hat{T}+1}) \right] \right\}, i \in N$$

subject to $x_{k+1} = f_k(x_k,u_k^1,u_k^2,\cdots,u_k^n) + \vartheta_k$,

where θ_k and ϑ_k are random variables.

Reference: *D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solutions For Randomly Furcating Stochastic Dynamic Games With Uncertain Horizon. International Game Theory Review, Vol. 16, 2014, pp.1440012.01-1440012.29.*

Theorem B4. (Nontransferable Individual Payoff in Continuous-time Stochastic Dynamic Cooperation)

$$-W_t^{(\alpha)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)\, W_{x^h x^\zeta}^{(\alpha)}(t,x) =$$

$$\max_{u_1,u_2}\left\{ \left(\sum_{j=1}^n \alpha^j\ g^j(t,x,u_1,u_2,\cdots,u_n)\right) exp\left[-\int_{t_0}^t r(y)dy\right] + W_x^{(\alpha)}(t,x) f(t,x,u^1,u^2,\cdots,u^n) \right\},$$

$W^{(\alpha)}(T,x) = \exp[-r(T-t_0)]\sum_{j=1}^n \alpha^j\ q^j\ (x)$,

$$-W_t^{(\alpha)i}(t,x) - \tfrac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)\, W_{x^h x^\zeta}^{(\alpha)i}(t,x) =$$

$$g^i\left[t,x,\psi_1^{(\alpha)}(t,x),\psi_2^{(\alpha)}(t,x),\cdots,\psi_n^{(\alpha)}(t,x)\right] exp\left[-\int_{t_0}^t r(y)dy\right]$$

$$+W_x^{(\alpha)i}(t,x) f\left[t,x,\psi_1^{(\alpha)}(t,x),\psi_2^{(\alpha)}(t,x),\cdots,\psi_n^{(\alpha)}(t,x)\right]$$ and

$W^{(\alpha)i}(T,x) = exp\left[-\int_{t_0}^T r(y)dy\right] q^i(x)$, for $i \in N$. ■

Origin and Application

Consider the problem of maximizing the joint weighted expected payoff of the players in a cooperative stochastic differential game with non-transferrable payoffs which maximizes

$$E_{t_0}\left\{ \sum_{j=1}^{n} \int_{t_0}^{T} \alpha^j\, g^j[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)]\, exp\left[-\int_{t_0}^{s} r(y)dy\right] ds + \sum_{j=1}^n\ exp\left[-\int_{t_0}^T r(y)dy\right] \alpha^j\, q^j(x(T)) \right\},$$

subject to

$dx(s) = f[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)]ds + \sigma[s,x(s)]dz(s)$, $x(t_0) = x_0$,

where $\sigma[s,x(s)]$ is a $n \times \Theta$ matrix and $z(s)$is a Θ-dimensional Wiener process and $\alpha = \left(\alpha^1,\alpha^2,\cdots,\alpha^n\right)$ for $\sum_{j=1}^n \alpha^j = 1$ is a set agreed-upon weights.

We denote a set of controls $\left\{u_i^{(\alpha)}(t) = \psi_i^{(\alpha)}(t,x), \text{ for } i \in N\right\}$ which provides an optimal solution to the above stochastic control problem and yields value functions $W^{(\alpha)}(t,x)\colon [t_0,T] \times R^n \to R$.

Then the individual payoff of player for $i \in N$ can be obtain as $W^{(\alpha)i}(t,x)\colon [t_0,T] \times R^n \to R$ as in Theorem B.4.

References: *D.W.K. Yeung: Nontransferable Individual Payoff Functions under Stochastic Dynamic Cooperation, International Game Theory Review, Vol. 6, 2004, pp. 281-289.*

D.W.K. Yeung: Nontransferable Individual Payoffs in Cooperative Stochastic Dynamic Games, International Journal of Algebra, Vol. 7, 2013, pp. 597-606.

Theorem B5. (Subgame-consistent Payoff Distribution Procedure (PDP) for Discrete-time Stochastic Dynamic Cooperation)

Consider the cooperative stochastic dynamic game Problem with player i payoff

$$E_{\theta_1,\theta_2,\cdots,\theta_T}\left\{\sum_{\zeta=1}^{T} g_\zeta^i[x_\zeta,u_\zeta^1,u_\zeta^2,\cdots,u_\zeta^n]\left(\frac{1}{1+r}\right)^{\zeta-1}+q_{T+1}^i(x_{T+!})\left(\frac{1}{1+r}\right)^T\right\},$$

and state dynamics $x_{k+1}=f_k(x_k,u_k^1,u_k^2,\cdots,u_k^n)+\theta_k$.

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^i\,(k,x_k^*)$ for player $i\in N$ in stage $k\in\kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_k^i(x_k^*)=(1+r)^{k-1}\Big\{\ \xi^i(k,x_k^*)$$
$$-E_{\theta_k}\Big(\ \xi^i\,[k+1,f_k(x_k^*,\psi_k^1(x_k^*),\psi_k^2(x_k^*),\cdots,\psi_k^n(x_k^*))+\theta_k]\ \Big)\ \Big\},$$

for $i\in N$,

given to player i at stage $k\in\kappa$, if $x_k^*\in X_k^*$ would lead to the realization of the imputation $\{\xi^i\ (k,x_k^*)$, for $i\in N$ and $k\in\kappa$ };

where

$\{\psi_k^i(x)$, for $k\in\kappa$ and $i\in N\}$ is a set of strategies that provides an optimal solution to the Problem B5 yielding functions $W(k,x)$, for $k\in K$, such that the following recursive relations are satisfied:

$$W(k,x)=\max_{u_k^1,u_k^2,\ldots,u_k^n}E_{\theta_k}\left\{\sum_{j=1}^{n} g_k^j[x,u_k^1,u_k^2,\cdots,u_k^n]\left(\frac{1}{1+r}\right)^{k-1}\right.$$
$$\left.+W[k+1,f_k(x,u_k^1,u_k^2,\cdots,u_k^n)+\theta_k]\ \right\},$$
$$W(T+1,x)=\sum_{j=1}^{n} q_{T+1}^j\,(x)\left(\frac{1}{1+r}\right)^T. \qquad ■$$

References: *D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Solutions for Cooperative Stochastic Dynamic Games. Journal of Optimization Theory and Applications, Vol. 145, 2010, pp. 579-596.*

Theorem B6. (Subgame-consistent PDP for Continuous-time Stochastic Dynamic Cooperation)

Consider the cooperative stochastic differential game Problem with the payoff of player $\boldsymbol{i}$ being

$$E_{t_0}\left\{\ \int_{t_0}^{T} g^i[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)]\,exp\left[-\int_{t_0}^{s} r(y)dy\right]ds\right.$$
$$\left.+\,exp\left[-\int_{t_0}^{T} r(y)dy\right]q^i\big(x(T)\big)\ \right\},\qquad \text{for } i\in N,$$

and state dynamics

$dx(s)=f[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)]ds+\sigma[s,x(s)]dz(s),\quad x(t_0)=x_0.$

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(s)i}(s,x_s^*)$ in current value at time s for player $i\in N$ in time $s\in[t_0,T]$ along the cooperative trajectory $\{x_s^*\}_{s=t_0}^T$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_i(s,x_s^*)=-\left[\xi_t^{(s)i}(t,x_t^*)\Big|_{t=s}\right]$$
$$-\left[\xi_{x_t^*}^{(s)i}(t,x_t^*)\Big|_{t=s}\right]f[s,x_s^*,\psi_1^*(s,x_s^*),\psi_2^*(s,x_s^*),\cdots,\psi_n^*(s,x_s^*)]$$
$$-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(s,x_s^*)\left[\xi_{x_t^h x_t^\zeta}^{(s)i}(t,x_t^*)\Big|_{t=s}\right],$$

for $i\in N$ and $x_s^*\in X_s^*$,

given to player i at time $s\in[t_0,T]$ would lead to the realization of the imputation $\{\xi^{(s)i}(s,x_s^*)$, for $i\in N$ and $s\in[t_0,T]\}$;

where

$\{\psi_i^*(s,x)$, for $i \in N$ and $s \in [t_0, T]$ is a set of strategies that provides an optimal solution to the Problem B6 yielding functions continuously twice differentiable functions $W(t,x)$:$[t_0,T] \times R^m \rightarrow R$, which satisfy the following partial differential equation:

$$-W_t(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)\, W_{x^h x^\zeta}(t,x) =$$

$$\max_{u_1, u_2, \dots, u_n} \left\{ \sum_{j=1}^{n} g^j[t,x,u_1, u_2, \dots, u_n]\, exp\left[-\int_{t_0}^{t} r(y)dy\right] + W_x(t,x) f[t,x,u_1,u_2,\cdots,u_n] \right\}, \text{ and}$$

$$W(T,x) = \sum_{j=1}^{n} q^j(x)\, exp\left[-\int_{t_0}^{T} r(y)dy\right]. \qquad \blacksquare$$

References: *D.W.K. Yeung and L. Petrosyan: Subgame Consistent Cooperative Solution in Stochastic Differential Games, Journal of Optimization Theory and Applications, Vol. 120, 2004, pp.651-666.*
D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Economic Optimization: An Advanced Cooperative Dynamic Game Analysis, Boston: Birkhäuser. ISBN 978-0-8176-8261-3, 395pp, 2012.

Theorem B7. (Subgame-consistent PDP for Random-horizon Dynamic Cooperation)

Consider the random-horizon cooperative dynamic game in which the objective of player i is

$$E\left\{ \sum_{k=1}^{\hat{T}} g_k^i[x_k, u_k^1, u_k^2, \cdots, u_k^n] + q_{\hat{T}+1}^i(x_{\hat{T}+1}) \right\}$$

$$= \sum_{\hat{T}=1}^{T} \theta_{\hat{T}} \left\{ \sum_{k=1}^{\hat{T}} g_k^i[x_k, u_k^1, u_k^2, \cdots, u_k^n] + q_{\hat{T}+1}^i(x_{\hat{T}+1}) \right\}, \text{ for } i \in N .$$

The players agree to maximize their joint payoff and share the cooperative gain according to the imputation $\xi^i(\tau, x_\tau^*)$ for player $i \in N$ in stage $\tau \in \kappa$ along the cooperative trajectory $\{x_\tau^*\}_{\tau=1}^T$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_\tau^i(x_\tau^*) = \xi^i(\tau, x_\tau^*) - \frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta} \xi^i(\tau+1, f_\tau[x_\tau, \psi_\tau^1(x_\tau), \psi_\tau^2(x_\tau), \cdots, \psi_\tau^n(x_\tau)])$$

$$- \frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta} q_{\tau+1}^i(f_\tau[x_\tau, \psi_\tau^1(x_\tau), \psi_\tau^2(x_\tau), \cdots, \psi_\tau^n(x_\tau)]), \quad \text{for } i \in N,$$

given to player i at stage $\tau \in \kappa$ would lead to the realization of the imputation $\xi^i(\tau, x_\tau^*)$ for player $i \in N$ in stage $\tau \in \kappa$;

where

$\{\psi_\tau^i(x)$, for $\tau \in \kappa$ and $i \in N\}$ is a set of strategies that provides a group optimal solution to the Problem 9 yielding functions $W(k,x)$, for $\tau \in \kappa$, such that the following recursive relations are satisfied:

$$V(T+1,x) = \sum_{j=1}^{n} q_{T+1}^j(x),$$

$$W(T,x) = \max_{u_T^1, u_T^2, \cdots, u_T^n} \left\{ \sum_{j=1}^{n} g_T^j[x, u_T^1, u_T^2, \cdots, u_T^n] + q_{T+1}[f_T(x, u_T, u_T^1, u_T^2, \cdots, u_T^n)] \right\},$$

$$W(\tau,x) = \max_{u_\tau^1, u_\tau^2, \cdots, u_\tau^n} \left\{ \sum_{j=1}^{n} g_\tau^j[x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n] + \frac{\theta_\tau}{\sum_{\zeta=\tau}^{T}\theta_\zeta} q_{\tau+1}^j[f_\tau(x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n)] \right]$$

$$+ \frac{\sum_{\zeta=\tau+1}^{T}\theta_\zeta}{\sum_{\zeta=\tau}^{T}\theta_\zeta} W[\tau+1, f_\tau(x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n)] \Big\}, \text{ for } \tau \in \{1,2,\cdots,T-1\}. \qquad \blacksquare$$

References: *D.W.K. Yeung and L. A. Petrosyan: Subgame Consistent Cooperative Solution of Dynamic Games with Random Horizon. Journal of Optimization Theory and Applications, Vol. 150, pp78-97, 2011.*

Theorem B8. (Subgame-consistent PDP for Discrete-time Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures)

Consider the randomly furcating cooperative stochastic dynamic game problem in which the objective that player i is

$E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\left\{ \sum_{k=1}^{T} g_k^i(x_k,u_k^1,u_k^2,\cdots,u_k^n;\theta_k) + q^i(x_{T+1}) \right\}$, for $i \in N$,
and the state dynamics is
$x_{k+1} = f_k(x_k,u_k^1,u_k^2,\cdots,u_k^n) + \vartheta_k$, and $x_1 = x^0$.

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(\sigma_k)}(k,x_k^*)$ along the cooperative trajectory given that $\theta_k^{\sigma_k}$ has occurred in stage k, for $\sigma_k \in \{1,2,\cdots,\eta_k\}$ and $k \in \{1,2,\cdots,T\}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_k^{(\sigma_k)i}(x_k^*) = \xi^{(\sigma_k)i}(k,x_k^*)$$
$$-E_{\vartheta_k}\left[\sum_{\sigma_{k+1}=1}^{\eta_{k+1}} \lambda_{k+1}^{\sigma_{k+1}} \left(\xi^{(\sigma_{k+1})i}\left[k+1, f_k\left(x_k^*, \psi_k^{(\sigma_k)*}(x_k^*)\right) + \vartheta_k\right] \right) \right],$$

for $i \in N$,
given to player i at stage $k \in \{1,2,\cdots,T\}$, if $\theta_k^{\sigma_k}$ occurs and $x_k^* \in X_k^*$, leads to the realization of the imputation $\xi^{(\sigma_k)}(k,x_k^*)$ for $k \in \{1,2,\cdots,T\}$;
where
where $\psi_t^{(\sigma_t)*}(x) = \{\psi_t^{(\sigma_t)1*}(x), \psi_t^{(\sigma_t)2*}(x), \cdots\cdots, \psi_t^{(\sigma_t)n*}(x)\}$, for $\sigma_t \in \{1,2,\cdots,\eta_t\}$ and $t \in \{1,2,\cdots,T\}$ is a set of strategies that provides a group optimal solution to Problem B8 yielding value functions $W^{(\sigma_t)}(t,x)$, for $\sigma_t \in \{1,2,\cdots,\eta_t\}$ and $t \in \{1,2,\cdots,T\}$, such that the following recursive relations are satisfied:

$$W^{(\sigma_T)}(T+1,x) = \sum_{j=1}^{n} q^j(x),$$
$$W^{(\sigma_T)}(T,x) = \max_{u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots, u_T^{(\sigma_T)n}} E_{\vartheta_T}\left\{ \sum_{j=1}^{n} g_T^j[x, u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots u_T^{(\sigma_T)n}; \theta_T^{\sigma_T}] \right.$$
$$\left. + W^{(\sigma_{T+1})}\left(T+1, f_T\left(x, u_T^{(\sigma_T)1}, u_T^{(\sigma_T)2}, \cdots, u_T^{(\sigma_T)n}\right) + \vartheta_T\right) \right\},$$
$$W^{(\sigma_t)}(t,x) = \max_{u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots, u_t^{(\sigma_t)n}} E_{\vartheta_t}\left\{ \sum_{j=1}^{n} g_t^j\left[x, u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots u_t^{(\sigma_t)n}; \theta_t^{\sigma_t}\right] \right.$$
$$\left. + \sum_{\sigma_{t+1}=1}^{\eta_{t+1}} \lambda_{t+1}^{\sigma_{t+1}} W^{(\sigma_{t+1})}\left[t+1, f_t\left(x, u_t^{(\sigma_t)1}, u_t^{(\sigma_t)2}, \cdots, u_t^{(\sigma_t)n}\right) + \vartheta_t\right] \right\},$$

for $\sigma_t \in \{1,2,\cdots,\eta_t\}$ and $t \in \{1,2,\cdots,T-1\}$. ■

Reference: *D.W.K. Yeung and L. A. Petrosyan: Subgame-consistent Cooperative Solutions in Randomly Furcating Stochastic Dynamic Games. Mathematical and Computer Modelling, Vol 57, pp.976–991, 2013.*

Theorem B9. Subgame-consistent PDP for Continuous-time Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures

Consider the randomly furcating cooperative stochastic dynamic game problem in which player $i \in N$ seeks to maximize the expected payoff:

$$E_{t_0}\left\{ \int_{t_0}^{t_1} g^{[i,\theta_0^0]}[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)]\, e^{-r(s-t_0)} ds \right.$$
$$\left. + \sum_{h=1}^{m} \sum_{a_h=1}^{\eta} \lambda_{a_h} \int_{t_h}^{t_{h+1}} g^{\left[i,\theta_{a_h}^h\right]}[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)] e^{-r(s-t_0)} + e^{-r(T-t_0)} q^i\big(x(T)\big) \right\},$$

and the state dynamics is

$$dx(s) = f[s,x(s),u_1(s),u_2(s),\cdots,u_n(s)]ds + \sigma[s,x(s)]dz(s), \qquad x(t_0) = x_0.$$

The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{i\left[\theta_{a_k}^k\right](k)\tau}(t,x_t^*)$, for $i \in N$, $\tau \in [t_k,t_{k+1}]$, $t \in [\tau,t_{k+1}]$, $k \in \{0,1,2,\cdots,m-1\}$, and $\theta_{a_k}^h \in \{\theta_1,\theta_2, \ldots,\theta_\eta\}$. A Payoff Distribution Procedure (PDP) with a payment equaling

$$B_i^{\left(\theta_{a_k}^k\right)k}(\tau) = -\left[\xi_t^{i\left[\theta_{a_k}^k\right](k)\tau}(t,x_t^*)|t=\tau\right]$$

$$-\left[\xi^{i[\theta^k_{\alpha_k}](k)\tau}_{x^*_t}(t,x^*_t)|t=\tau\right] \quad f\left[\tau,x^*_\tau,\psi^{(k)\theta^k_{\alpha_k}}_1(\tau,x^*_\tau),\psi^{(k)\theta^k_{\alpha_k}}_2(\tau,x^*_\tau),\cdots,\psi^{(k)\theta^k_{\alpha_k}}_n(\tau,x^*_\tau)\right]$$
$$-\tfrac{1}{2}\textstyle\sum_{h,\zeta=1}^{n}\Omega^{h\zeta}(\tau,x^*_\tau)\left[\xi^{i[\theta^k_{\alpha_k}](k)\tau}_{x^h_t x^\zeta_t}(t,x^*_t)\Big|_{t=\tau}\right] \;,$$
for $i\in N$ and $k\in\{1,2,\cdots,m\}$,
given to player i at time $\tau\in[t_k,t_{k+1}]$ contingent upon $\theta^k_{\alpha_k}\in\{\theta_1,\theta_2,\ \ldots,\theta_\eta\}$has occurred at time t_k, leads to the realization of the imputation $\xi^{i[\theta^k_{\alpha_k}](k)\tau}(t,x^*_t)$, for $i\in N$, $\tau\in[t_k,t_{k+1}]$, $t\in[\tau,t_{k+1}]$, $k\in\{0,1,2,\cdots,m-1\}$, and $\theta^h_{\alpha_k}\in\{\theta_1,\theta_2,\ \ldots,\theta_\eta\}$.
where
$\{u^{(m)\theta^m_{\alpha_m}}_i(t)=\psi^{(m)\theta^m_{\alpha_m}}_i(t,x)$, for $t\in[t_m,T]$; $u^{(k)\theta^k_{\alpha_k}}_i(t)=\psi^{(k)\theta^k_{\alpha_k}}_i(t,x)$, for $t\in[t_k,t_{k+1}]$, $k\in\{0,1,2,\cdots,m-1\}$ and $i\in N\}$, contingent upon the events $\theta^m_{\alpha_m}$and $\theta^k_{\alpha_k}$ is a set of controls that provides a group optimal solution for the game Problem 11 yielding continuously differentiable functions $W^{[\theta^m_{\alpha_m}](m)}(t,x):[t_m,T]\times R^\kappa\to R$ and $W^{[\theta^k_{\alpha_k}](k)}(t,x):[t_k,t_{k+1}]\times R^\kappa\to R$ for $k\in\{0,1,2,\cdots,m-1\}$ which satisfy the following partial differential equations:
$$-W^{[\theta^m_{\alpha_m}](m)}_t(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega^{h\zeta}(t,x)\,W^{[\theta^m_{\alpha_m}](m)}_{x^h\,x^\zeta}(t,x)$$
$$=\max_{u_1^{\theta_{\alpha_m}},u_2^{\theta_{\alpha_m}},\cdots,u_n^{\theta_{\alpha_m}}} E_{\vartheta_T}\Big\{\ \textstyle\sum_{j=1}^{n}\ g^{[j,\theta^m_{\alpha_m}}[t,x(t),u^{(m)\theta^m_{\alpha_m}}_1,u^{(m)\theta^m_{\alpha_m}}_2,\cdots,u^{(m)\theta^m_{\alpha_m}}_n\ ]e^{-r(t-t_\tau)}$$
$$+W^{[\theta^m_{\alpha_m}](m)}_x(t,x)f\left[t,x,u^{(m)\theta^m_{\alpha_m}}_1,u^{(m)\theta^m_{\alpha_m}}_2,\cdots,u^{(m)\theta^m_{\alpha_m}}_n\right]\quad\Big\},\text{ and}$$
$$W^{[\theta^m_{\alpha_m}](m)}(T,x)=e^{-r(T-t_0)}\textstyle\sum_{j=1}^{n}q^j(x);$$
$$-W^{[\theta^k_{\alpha_k}](k)}_t(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{n}\Omega^{h\zeta}(t,x)\,W^{[\theta^k_{\alpha_k}](k)}_{x^h\,x^\zeta}(t,x)$$
$$=\max_{u_1^{\theta_{\alpha_k}},u_2^{\theta_{\alpha_k}},\cdots,u_n^{\theta_{\alpha_k}}}\Big\{\ \sum_{j=1}^{n}\ g^{[j,\theta^k_{\alpha_k}}[t,x,u^{(k)\theta^k_{\alpha_k}}_1,u^{(k)\theta^k_{\alpha_k}}_2,\cdots,u^{(k)\theta^k_{\alpha_k}}_n\ \ e^{-r(t-t_k)}$$
$$+W^{[\theta^k_{\alpha_k}](k)}_x(t,x)f\left[t,x,u^{(k)\theta^k_{\alpha_k}}_1,u^{(k)\theta^k_{\alpha_k}}_2,\cdots,u^{(k)\theta^k_{\alpha_k}}_n\right]\quad\Big\},\text{ and}$$
$$W^{[\theta^k_{\alpha_k}](k)}(t_{k+1},x)=\textstyle\sum_{a=1}^{\eta}\lambda_a\,W^{[\theta^{k+1}_a](k)}(t_{k+1},x),\quad\text{for }k\in\{0,1,2,\cdots,m-1\}.\qquad\blacksquare$$
References: *L. A. Petrosyan and D.W.K. Yeung: Subgame-consistent Cooperative Solutions in Randomly-furcating Stochastic Differential Games, International Journal of Mathematical and Computer Modelling (Special Issue on Lyapunov's Methods in Stability and Control), Vol. 45, June 2007, pp.1294-1307.*
L. A. Petrosyan and D.W.K. Yeung: Subgame Consistent Cooperation – A Comprehensive Treaties, Springer 2016.

Theorem B10. (Subgame-consistent PDP for Random-horizon Stochastic Dynamic Cooperation under Uncertainty in Payoff Structures)

Consider the uncertain horizon randomly furcating cooperative stochastic dynamic game problem in which the objective that player $i\in N$ is
$$E_{\theta_1,\theta_2,\cdots,\theta_T;\vartheta_1,\vartheta_2,\cdots,\vartheta_T}\Big\{\ \ \textstyle\sum_{\hat T=1}^{T}\varpi_{\hat T}\Big[\ \ \sum_{k=1}^{\hat T}\ \ g^i_k[x_k,u^1_k,u^2_k,\cdots,u^n_k;\theta_k]+q^i\ (x_{\hat T+1})\ \ \Big]\ \ \Big\},$$
and the state dynamics is
$$x_{k+1}=f_k(x_k,u^1_k,u^2_k,\cdots,u^n_k)+\vartheta_k,\quad x_1=x^0\,.$$
The players agree to maximize their joint expected payoff and share the cooperative gain according to the imputation $\xi^{(\sigma_k)}(k,x^*_k)=\left[\xi^{(\sigma_k)1}(k,x^*_k),\xi^{(\sigma_k)2}(k,x^*_k),\cdots,\xi^{(\sigma_k)n}(k,x^*_k)\right]$ along the cooperative trajectory given that $\theta^{\sigma_k}_k$ has occurred in stage k, for $\sigma_k\in\{1,2,\cdots,\eta_k\}$ and $k\in\{1,2,\cdots,T\}$. A Payoff Distribution Procedure (PDP) with a payment equaling
$$B^{(\sigma_k)i}_k(x^*_k)=\xi^{(\sigma_k)i}(k,x^*_k)-E_{\vartheta_k}\Big\{\ \ \frac{\varpi_k}{\sum_{\zeta=k}^{T}\varpi_\zeta}q^i_{k+1}\left[f_k\left(x^*_k,\psi^{(\sigma_k)*}_k(x^*_k)\right)+\vartheta_k\right]$$

$$+\frac{\sum_{\mu=k+1}^{T}\varpi_\mu}{\sum_{\zeta=k}^{T}\varpi_\zeta}\sum_{\sigma_{k+1}=1}^{\eta_k}\lambda_{k+1}^{\sigma_{k+1}}\,\xi^{(\sigma_{k+1})i}[k+1, f_k(x_k^*, \psi_k^{(\sigma_k)*}(x_k^*)) + \vartheta_k] \quad \Big\},$$

given to player $i \in N$ at stage $k \in \{1,2,\cdots,T\}$ if $\theta_k^{\sigma_k} \in \{\theta_k^1, \theta_k^2, \cdots, \theta_k^{\eta_k}\}$ occurs would lead to the realization of the imputation:

$\xi^{(\sigma_k)}(k, x_k^*) = \left[\xi^{(\sigma_k)1}(k, x_k^*), \xi^{(\sigma_k)2}(k, x_k^*), \cdots, \xi^{(\sigma_k)n}(k, x_k^*)\right]$, for $\sigma_k \in \{1,2,\cdots,\eta_k\}$ and $k \in \{1,2,\cdots,T\}$;

where

$\psi_k^{(\sigma_k)*}(x_k^*) = \left[\psi_k^{(\sigma_k)1*}(x_k^*), \psi_k^{(\sigma_k)2*}(x_k^*), \cdots, \psi_k^{(\sigma_k)n*}(x_k^*)\right]$, for $k \in \kappa$ and $i \in N$ is a set of controls that provides a group optimal solution to the Problem B10 yielding functions $W^{(\sigma_t)}(t,x)$, for $\sigma_t \in \{1,2,\cdots,\eta_t\}$ and $t \in \{1,2,\cdots,T\}$, such that the following recursive relations are satisfied:

$$W^{(\sigma_{T+1})}(T+1, x) = \sum_{j=1}^{n} q_{T+1}^j(x),$$

$$W^{(\sigma_T)}(T, x) = \max_{u_T^1, u_T^2, \cdots, u_T^n} E_{\vartheta_T}\Big\{ \sum_{j=1}^{n} g_T^j\big(x, u_T^1, u_T^2, \cdots, u_T^n; \theta_T^{\sigma_T}\big)$$
$$+W^{(\sigma_{T+1})}[T+1, f_T(x, u_T^1, u_T^2, \cdots, u_T^n) + \vartheta_T] \quad \Big\},$$

$$W^{(\sigma_\tau)}(\tau, x) = \max_{u_\tau^1, u_\tau^2, \cdots, u_\tau^n} E_{\vartheta_\tau}\Big\{ \sum_{j=1}^{n} g_\tau^j\big(x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n; \theta_\tau^{\sigma_\tau}\big)$$
$$+\sum_{j=1}^{n} \frac{\varpi_\tau}{\sum_{\zeta=\tau}^{T}\varpi_\zeta} q_{\tau+1}^j[f_\tau(x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n) + \vartheta_\tau]$$
$$+\frac{\sum_{\zeta=\tau+1}^{T}\varpi_\zeta}{\sum_{\zeta=\tau}^{T}\varpi_\zeta}\sum_{\sigma_{\tau+1}=1}^{\eta_{\tau+1}}\lambda_{\tau+1}^{\sigma_{\tau+1}}\,W^{(\sigma_{\tau+1})}[\tau+1, f_\tau(x, u_\tau^1, u_\tau^2, \cdots, u_\tau^n) + \vartheta_\tau] \quad \Big\}$$

for $\tau \in \{1,2,\cdots,T-1\}$. ■

Reference: *D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solutions For Randomly Furcating Stochastic Dynamic Games With Uncertain Horizon. International Game Theory Review, Vol. 16, pp.1440012.01-1440012.29, 2014.*

Theorem B11. (Subgame-consistent Solution Mechanism for Dynamic Cooperation under Non-transferrable Payoffs (NTU))

Consider the non-transferrable payoff/utility (NTU) cooperative dynamic game problem in which the payoff of player $i \in N$ is

$$\sum_{k=1}^{T} g_k^i\,(x_k, u_k^1, u_k^2, \cdots, u_k^n) + q^i\,(x_{T+1}),$$

and the state dynamics is $x_{k+1} = f_k(x_k, u_k^1, u_k^2, \cdots, u_k^n)$.

The players agree to use a set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \cdots, \hat{\alpha}_k^n)$, for $k \in \kappa\}$ for joint maximization of the weighted joint payoff so that the imputation $\xi^i\,(k, x_k^*)$ for player $i \in N$ in stage $k \in \kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^{T}$ can be achieved.

A set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \cdots, \hat{\alpha}_k^n)$, for $k \in \kappa\}$ and a set of strategies $\{\psi_k^{(\hat{\alpha}_k)i}(x)$, for $k \in \kappa$ and $i \in N\}$ provides a subgame consistent solution to the above NTU cooperative dynamic game if there exist functions $W^{(\hat{\alpha}_k)}(k,x)$ and $W^{(\hat{\alpha}_k)i}(k,x)$, for $i \in N k \in \kappa$, which satisfy the following recursive relations:

$$W^{(\hat{\alpha}_{T+1})i}(T+1, x) = q^i\,(x_{T+1}),$$

$$W^{(\hat{\alpha}_k)}(k, x) = \max_{u_k^1, u_k^2, \cdots, u_k^n}\Big\{ \sum_{j=1}^{n} \hat{\alpha}^j\; g_k^j(x, u_k^1, u_k^2, \cdots, u_k^n)$$
$$+\sum_{j=1}^{n} \hat{\alpha}_k^j\, W^{(\hat{\alpha}_{k+1})j}[k+1, f_k(x, u_k^1, u_k^2, \cdots, u_k^n)] \quad \Big\};$$

$$W^{(\hat{\alpha}_k)i}(k, x) = g_k^j\left(x, \psi_k^{(\hat{\alpha}_k)1}(x), \psi_k^{(\hat{\alpha}_k)2}(x), \cdots, \psi_k^{(\hat{\alpha}_k)n}(x)\right)$$
$$+W^{(\hat{\alpha}_{k+1})i}\left[k+1, f_k\left(x, \psi_k^{(\hat{\alpha}_k)1}(x), \psi_k^{(\hat{\alpha}_k)2}(x), \cdots, \psi_k^{(\hat{\alpha}_k)n}(x)\right)\right],$$

for $i \in N$ and $k \in \kappa$;

and $\xi^i\,(k, x_k^*)$ is the imputation for player $i \in N$ in stage $k \in \kappa$, where the value function $W^{(\hat{\alpha}_k)i}(k,x)$ is the payoff for player $i \in N$ in stage $k \in \kappa$ under cooperation. ■

Reference: *D.W.K. Yeung and L.A. Petrosyan: Subgame Consistent Cooperative Solution for NTU Dynamic Games via Variable Weights, forthcoming in Automatica, 2015.*

Theorem B12. (Subgame-consistent Solution Mechanism for Stochastic Dynamic Cooperation under Non-transferrable Payoffs (NTU))

Consider the non-transferrable payoff/utility (NTU) cooperative stochastic dynamic game problem in which he payoff of player $i \in N$ is

$$E_{\theta_1,\theta_2,\cdots,\theta_T}\left\{ \sum_{\zeta=1}^{T} g_\zeta^i[x_\zeta, u_\zeta^1, u_\zeta^2, \cdots, u_\zeta^n, x_{\zeta+1}] + q^i\ (x_{T+1}) \right\},$$

and the state dynamics is $x_{k+1} = f_k(x_k, u_k^1, u_k^2, \cdots, u_k^n) + G_k(x_k)\theta_k$.

The players agree to use a set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \cdots, \hat{\alpha}_k^n)$, for $k \in \kappa\}$ for joint maximization of the expected weighted joint payoff so that the imputation $\xi^i\ (k, x_k^*)$ for player $i \in N$ in stage $k \in \kappa$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$ can be achieved.

A set of payoff weights $\{\hat{\alpha}_k = (\hat{\alpha}_k^1, \hat{\alpha}_k^2, \cdots, \hat{\alpha}_k^n)$, for $k \in \kappa\}$ and a set of strategies $\{\psi_k^{(\hat{\alpha}_k)i}(x)$, for $k \in \kappa$ and $i \in N\}$ provides a subgame consistent solution to the above NTU cooperative dynamic game B12 if there exist functions $W^{(\hat{\alpha}_k)}(k,x)$ and $W^{(\hat{\alpha}_k)i}(k,x)$, for $i \in N k \in \kappa$, which satisfy the following recursive relations:

$$W^{(\hat{\alpha}_{T+1})i}(T+1,x) = q^i\ (x_{T+1}),$$

$$W^{(\hat{\alpha}_k)}(k,x) = \max_{u_k^1,u_k^2,\cdots,u_k^n}\left\{ E_{\theta_k}\left[\sum_{j=1}^{n} \hat{\alpha}^j\ g_k^j(x, u_k^1, u_k^2, \cdots, u_k^n) \right.\right.$$
$$\left.\left. + \sum_{j=1}^{n} \hat{\alpha}_k^j\, W^{(\hat{\alpha}_{k+1})j}[k+1, f_k(x, u_k^1, u_k^2, \cdots, u_k^n) + G_k(x)\theta_k] \right] \right\};$$

$$W^{(\hat{\alpha}_k)i}(k,x) = E_{\theta_k}\left\{ g_k^j\left(x, \psi_k^{(\hat{\alpha}_k)1}(x), \psi_k^{(\hat{\alpha}_k)2}(x), \cdots, \psi_k^{(\hat{\alpha}_k)n}(x)\right) \right.$$
$$\left. + W^{(\hat{\alpha}_{k+1})i}\left[k+1, f_k\left(x, \psi_k^{(\hat{\alpha}_k)1}(x), \psi_k^{(\hat{\alpha}_k)2}(x), \cdots, \psi_k^{(\hat{\alpha}_k)n}(x)\right) + G_k(x)\theta_k\right] \right\},$$

for $i \in N$ and $k \in \kappa$;

and $\xi^i\ (k, x_k^*)$ is the imputation for player $i \in N$ in stage $k \in \kappa$,

where the value function $W^{(\hat{\alpha}_k)i}(k,x)$ is the expected payoff for player $i \in N$ in stage $k \in \kappa$ under cooperation. ■

Reference: *D.W.K. Yeung and L.A. Petrosyan: On Subgame Consistent Solution for NTU Cooperative Stochastic Dynamic Games, paper presented at European Meeting on Game Theory (SING11-GTM2015) at St Petersburg, July 8-10, 2015.*

Theorem B13. Hamilton-Jacobi-Bellman Equations for Dynamic Games with Durable Controls

Let $\left(\underline{u}_k^{**}, \bar{\underline{u}}_k^{**}\right)$ be the set of feedback Nash equilibrium strategies and $V^i\left(k, x; \bar{\underline{u}}_{k-}^{**}\right)$ be the feedback Nash equilibrium payoff of player i at stage k in the non-cooperative dynamic game where the payoff of player $i \in N$is

$$\sum_{k=1}^{T} g_k^i\left(x_k, \underline{u}_k, \bar{\underline{u}}_k; \bar{\underline{u}}_{k-}\right)\delta_1^k + q_{T+1}^i\left(x_{T+1}; \bar{\underline{u}}_{(T+1)-}\right)\delta_1^{T+1},$$

and the state dynamics is

$$x_{k+1} = f_k\left(x_k, \underline{u}_k, \bar{\underline{u}}_k; \bar{\underline{u}}_{k-}\right), \qquad x_1 = x_1^0,$$

for $k \in \{1,2,\cdots,T\}$.

Then the function $V^i\left(k, x; \bar{\underline{u}}_{k-}^{**}\right)$ satisfies the following recursive equations:

$$V^i\left(T+1, x; \bar{\underline{u}}_{(T+1)-}^{**}\right) = q_{T+1}^i\left(x; \bar{\underline{u}}_{(T+1)-}^{**}\right)\delta_1^{T+1};$$

$$V^i\left(k,x;\underline{u}_{k-}^{**}\right)=\max_{u_k^i,\bar{u}_k^i}\Big\{\quad g_k^i\big(x,u_k^i,\bar{u}_k^i,\underline{u}_k^{**(\neq i)},\underline{\bar{u}}_k^{**(\neq i)};\underline{\bar{u}}_{k-}^{**}\big)\delta_1^k$$

$$+V^i\left[k+1,f_k\big(x,u_k^i,\bar{u}_k^i,\underline{u}_k^{**(\neq i)},\underline{\bar{u}}_k^{**(\neq i)};\underline{\bar{u}}_{k-}^{**}\big);\bar{u}_k^i,\underline{\bar{u}}_k^{**(\neq i)},\underline{\bar{u}}_{(k+1)-}^{**}\cap\underline{\bar{u}}_{k-}^{**}\right]\quad\Big\},$$

for $k\in\{1,2,\cdots,T\}$ and $i\in N$,

where $\underline{u}_k^{(\neq i)**}=\left(u_k^{1**},u_k^{2**},\cdots,u_k^{i-1**},u_k^{i+1**},\cdots,u_k^{n**}\right)$, and

$$\underline{\bar{u}}_k^{(\neq i)**}=\left(\bar{u}_k^{1**},\bar{u}_k^{2**},\cdots,\bar{u}_k^{i-1**},\bar{u}_k^{i+1**},\cdots,\bar{u}_k^{n**}\right).\quad\blacksquare$$

***References*: *D.W.K. Yeung, L.A. Petrosyan (2020): Cooperative Dynamic Games with Durable Controls: Theory and Application, Dynamic Games and Applications, Doi:10.1007/s13235-019-00336-w. 9(2), 550-567, 2019, https://doi.org/10.1007/s13235-018-0266-6.*

D.W.K. Yeung, L.A. Petrosyan (2019): Cooperative Dynamic Games with Control Lags, Dynamic Games and Applications, 9(2), 550-567, https://doi.org/10.1007/s13235-018-0266-6.

Theorem B14. Subgame-consistent PDP for Cooperative Dynamic Games with Durable Controls

Consider the cooperative game in which the payoff of player $i\in N$is

$$\sum_{k=1}^T g_k^i\left(x_k,\underline{u}_k,\underline{\bar{u}}_k;\underline{\bar{u}}_{k-}\right)\delta_1^k+q_{T+1}^i\big(x_{T+1};\underline{\bar{u}}_{(T+1)-}\big)\delta_1^{T+1},$$

and the state dynamics is

$$x_{k+1}=f_k\big(x_k,\underline{u}_k,\underline{\bar{u}}_k;\underline{\bar{u}}_{k-}\big),\qquad x_1=x_1^0.$$

The agreed-upon imputation $\xi\big(k,x_k^*;\underline{\bar{u}}_{k-}^*\big)$, for $k\in\{1,2,\cdots,T\}$ along the cooperative trajectory $\{x_k^*\}_{k=1}^T$, can be realized by a payment

$$\beta_k^i(x_k^*;\underline{\bar{u}}_{k-}^*)=(\delta_1^k)^{-1}\Big[\quad\xi^i\left(k,x_k^*;\underline{\bar{u}}_{k-}^*\right)-\xi^i\Big(\quad k+1,f_k(x_k^*,\underline{u}_k^*,\underline{\bar{u}}_k^*;\underline{\bar{u}}_{k-}^*);\underline{\bar{u}}_{(k+1)-}^*\quad\Big)\quad\Big]$$

given to player $i\in N$ at stage $k\in\{1,2,\cdots,T\}$. $\blacksquare$

***References*: *D.W.K. Yeung, L.A. Petrosyan (2020): Cooperative Dynamic Games with Durable Controls: Theory and Application, Dynamic Games and Applications, Doi:10.1007/s13235-019-00336-w. 9(2), 550-567, 2019, https://doi.org/10.1007/s13235-018-0266-6.*

D.W.K. Yeung, L.A. Petrosyan (2019): Cooperative Dynamic Games with Control Lags, Dynamic Games and Applications, 9(2), 550-567, https://doi.org/10.1007/s13235-018-0266-6.

Theorem B15. Hamilton-Jacobi-Bellman Equations for Dynamic Games under Random Horizon and Durable Controls

Consider the random horizon dynamic game in which player $i\in N$ seeks to maximize

$$E\Big\{\quad\sum_{k=1}^{\hat{T}}\quad g_k^i\big(x_k,\underline{u}_k,\underline{\bar{u}}_k;\underline{\bar{u}}_{k-}\big)\delta_1^k+q_{\hat{T}+1}^i\big(x_{\hat{T}+1};\underline{\bar{u}}_{(\hat{T}+1)-}\big)\delta_1^{\hat{T}+1}\quad\Big\}$$

$$=\sum_{\hat{T}=1}^T\theta_{\hat{T}}\Big\{\quad\sum_{k=1}^{\hat{T}}\quad g_k^i\big(x_k,\underline{u}_k,\underline{\bar{u}}_k;\underline{\bar{u}}_{k-}\big)\delta_1^k+q_{\hat{T}+1}^i\big(x_{\hat{T}+1};\underline{\bar{u}}_{(\hat{T}+1)-}\big)\delta_1^{\hat{T}+1}\quad\Big\},\quad i\in N,$$

subject to the dynamics

$$x_{k+1}=f_k\big(x_k,\underline{u}_k,\underline{\bar{u}}_k;\underline{\bar{u}}_{k-}\big),\qquad x_1=x_1^0,$$

where the state $x_\tau=x$ with previously executed controls $\underline{\bar{u}}_{\tau-}$ by all players.

Let $\{\underline{u}_\tau^{**}, \bar{\underline{u}}_\tau^{**}\}$ denote the set of the players' feedback Nash equilibrium strategies and $V^i\left(\tau, x; \bar{\underline{u}}_{\tau-}^{**}\right)$ denote the feedback Nash equilibrium payoff of player i in the above non-cooperative game, then the function $V^i\left(\tau, x; \bar{\underline{u}}_{\tau-}^{**}\right)$ satisfies the following system of recursive equations

$$V^i\left(T+1, x; \bar{\underline{u}}_{(T+1)-}^{**}\right) = q_{T+1}^i\left(x; \bar{\underline{u}}_{(T+1)-}^{**}\right)\delta_1^{T+1},$$

$$V^i\left(T, x; \bar{\underline{u}}_{T-}^{**}\right) = \max_{u_T^i, \bar{u}_T^i}\Big\{ \quad g_T^i\left(x, u_T^i, \bar{u}_T^i, \underline{u}_T^{(\neq i)**}, \bar{\underline{u}}_T^{(\neq i)**}; \bar{\underline{u}}_{T-}^{**}\right)\delta_1^T$$

$$+q_{T+1}^i\left[f_T\left(x, u_T^i, \bar{u}_T^i, \underline{u}_T^{(\neq i)**}, \bar{\underline{u}}_T^{(\neq i)**}; \underline{u}_{T-}^{**}\right); \bar{u}_T^i, \bar{\underline{u}}_T^{(\neq i)**}, \bar{\underline{u}}_{(T+1)-}^{**} \cap \bar{\underline{u}}_{T-}^{**}\right]\delta_1^{T+1} \quad \Big\},$$

$$V^i\left(\tau, x; \bar{\underline{u}}_{\tau-}^{**}\right) = \max_{u_\tau^i, \bar{u}_\tau^i}\Big\{ \quad g_\tau^i\left(x, u_\tau^i, \bar{u}_\tau^i, \underline{u}_\tau^{(\neq i)**}, \bar{\underline{u}}_\tau^{(\neq i)**}; \bar{\underline{u}}_{\tau-}^{**}\right)\delta_1^\tau$$

$$+\frac{\theta_\tau}{\sum_{\zeta=\tau}^T \theta_\zeta} q_{\tau+1}^i\left[f_\tau\left(x, u_T^i, \bar{u}_T^i, \underline{u}_T^{(\neq i)**}, \bar{\underline{u}}_T^{(\neq i)**}; \underline{u}_{\tau-}^{**}\right); \bar{u}_\tau^i, \bar{\underline{u}}_\tau^{(\neq i)**}, \bar{\underline{u}}_{(\tau+1)-}^{**} \cap \bar{\underline{u}}_{\tau-}^{**}\right]\delta_1^{\tau+1}$$

$$+\frac{\sum_{\zeta=\tau+1}^T \theta_\zeta}{\sum_{\zeta=\tau}^T \theta_\zeta} V^i\left[\tau+1, f_\tau\left(x, u_\tau^i, \bar{u}_\tau^i, \underline{u}_\tau^{(\neq i)**}, \bar{\underline{u}}_\tau^{(\neq i)**}; \underline{u}_{\tau-}^{**}\right); \bar{u}_\tau^i, \bar{\underline{u}}_\tau^{(\neq i)**}, \bar{\underline{u}}_{(\tau+1)-}^{**} \cap \bar{\underline{u}}_{\tau-}^{**}\right] \quad \Big\},$$

for $\tau \in \{1, 2, \cdots, T-1\}$ and $i \in N$,

where $\underline{u}_\tau^{(\neq i)**} = \underline{u}_\tau^{**} \backslash u_\tau^{i**}$ and $\bar{\underline{u}}_\tau^{(\neq i)**} = \bar{\underline{u}}_\tau^{**} \backslash \bar{u}_\tau^{i**}$. ■

References: *D.W.K. and Petrosyan, L.A.: Generalized dynamic games with durable strategies under uncertain planning horizon, Journal of Computational and Applied Mathematics, 395 (2021) 113595.*

Theorem B16. Subgame-consistent PDP for Cooperative Dynamic Games under Random Horizon and Durable Controls

Consider the random horizon cooperative dynamic game in which the payoff of player $i \in N$ is

$$E\Big\{ \quad \sum_{k=1}^{\hat{T}} \quad g_k^i\left(x_k, \underline{u}_k, \bar{\underline{u}}_k; \bar{\underline{u}}_{k-}\right)\delta_1^k + q_{\hat{T}+1}^i\left(x_{\hat{T}+1}; \bar{\underline{u}}_{(\hat{T}+1)-}\right)\delta_1^{\hat{T}+1} \quad \Big\}$$

$$= \sum_{\hat{T}=1}^T \theta_{\hat{T}}\Big\{ \quad \sum_{k=1}^{\hat{T}} \quad g_k^i\left(x_k, \underline{u}_k, \bar{\underline{u}}_k; \bar{\underline{u}}_{k-}\right)\delta_1^k + q_{\hat{T}+1}^i\left(x_{\hat{T}+1}; \bar{\underline{u}}_{(\hat{T}+1)-}\right)\delta_1^{\hat{T}+1} \quad \Big\}, \quad i \in N,$$

and the dynamics is

$$x_{k+1} = f_k\left(x_k, \underline{u}_k, \bar{\underline{u}}_k; \bar{\underline{u}}_{k-}\right), \qquad x_1 = x_1^0.$$

The players agree to cooperate and the agreed-upon imputation is $\xi\left(k, x_\tau^*; \bar{\underline{u}}_{\tau-}^*\right)$, for $k \in \{1, 2, \cdots, T\}$ along the cooperative trajectory $\{x_\tau^*\}_{\tau=1}^T$. The agreed-upon imputation $\xi\left(k, x_\tau^*; \bar{\underline{u}}_{\tau-}^*\right)$ can be realized by a payment equaling

$$\beta_\tau^i(x_\tau^*; \bar{\underline{u}}_{\tau-}^*) = (\delta_1^\tau)^{-1}\Big(\quad \xi^i\ (\tau, x_\tau^*; \bar{\underline{u}}_{\tau-}^*) - \frac{\theta_\tau}{\sum_{\zeta=\tau}^T \theta_\zeta} q_{\tau+1}^i(x_{\tau+1}^*; \bar{\underline{u}}_{(\tau+1)-}^*)\delta_1^{\tau+1}$$

$$-\frac{\sum_{\zeta=\tau+1}^T \theta_\zeta}{\sum_{\zeta=\tau}^T \theta_\zeta}\xi^i\ \left(\tau+1, x_{\tau+1}^*; \bar{\underline{u}}_{(\tau+1)-}^*\right) \quad \Big),$$

given to player $i \in N$ at stage $\tau \in \{1, 2, \cdots, T\}$. ■

References: *D.W.K. and Petrosyan, L.A.: Generalized dynamic games with durable strategies under uncertain planning horizon, Journal of Computational and Applied Mathematics, 395 (2021) 113595.*

Part C: Identities and Equations in Economics

C1. Inter-temporal Roy's Identity

Consider the consumer problem in which the consumer maximizes his inter-temporal utility

$$u^1\left(x_1^1, x_1^2, \cdots, x_1^{n_1}\right) + \sum_{k=2}^{T} \delta_2^k\, u^k\left(x_k^1, x_k^2, \cdots, x_k^{n_k}\right)$$
$$= u^1(x_1) + \sum_{k=2}^{T} \delta_2^k\, u^k(x_k) = \sum_{k=1}^{T} \delta_1^k\, u^k(x_k)$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{\eta_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{\eta_k} p_k^h x_k^h) + Y_{k+1}, \quad W_1 = W_1^0,$$

where
$x_k = \left(x_k^1, x_k^2, \cdots, x_k^{n_k}\right)$ is the vector of quantities of goods consumed in period k, $p_k = \left(p_k^1, p_k^2, \cdots, p_k^{n_k}\right)$ is price vector, r is the interest rate, and Y_k is the income that the consumer will receive in period k.

The Roy's Identity of the above inter-temporal utility problem can be formulated as:

$$\frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^j} \div \frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}$$
$$\equiv -(1+r)^{-(h-\ell)} \phi_h^j(W_h^0, p_h, p_{h+1}, \cdots, p_T);$$

or in an alternative form

$$\frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^j} \div \delta_{\ell+1}^h \frac{\partial v^h\,(W_h^0, p_h, p_{h+1}, \cdots, p_T)}{\partial W_h^0}$$
$$\equiv -\phi_h^j(W_h^0, p_h, p_{h+1}, \cdots, p_T);$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell, \ell+1, \cdots, T\}$ and $j \in \{1,2,\cdots,n_h\}$,
where

$$W_\ell = W_\ell^0,$$
$$W_{\ell+1}^0 = (1+r)(W_\ell^0 - p_\ell \phi_\ell) + Y_{\ell+1},$$
$$W_{\ell+2}^0 = (1+r)(W_{\ell+1}^0 - p_{\ell+1}\phi_{\ell+1}) + Y_{\ell+2},$$
$$\vdots \qquad\qquad \vdots$$
$$W_h^0 = (1+r)(W_{h-1}^0 - p_{h-1}\phi_{h-1}) + Y_h. \qquad \blacksquare$$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141

C2. Inter-temporal Roy's Identity under Stochastic Life-span

Consider the consumer problem in C1 with the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1, \gamma_2, \cdots, \gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau, \tau+1, \cdots, T$ becomes respectively

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}, \cdots, \frac{\gamma_T}{\sum_{\zeta=\tau}^{T} \gamma_\zeta}.$$

The consumer maximizes his expected inter-temporal utility

$$\sum_{\hat{T}=1}^{T} \gamma_{\hat{T}} \sum_{k=1}^{\hat{T}} \delta_1^k\, u^k(x_k),$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\left(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\right) + Y_{k+1}, \qquad W_1 = W_1^0.$$

The Roy's identity of the above inter-temporal utility problem is formulated as:

$$\frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^j} \div \frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}$$
$$\equiv -(1+r)^{-(h-\ell)} \phi_h^j(W_h^0, p_h, p_{h+1}, \cdots, p_T);$$

or in an alternatively form:

$$\frac{\partial v^{\ell}\,(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial p_h^{j}} \div \delta_{\ell+1}^{h} \frac{\partial v^{h}\,(W_h^{0}, p_h, p_{h+1}, \cdots, p_T)}{\partial W_h^{0}}$$

$$\equiv -\frac{\sum_{\zeta=h}^{T} \gamma_{\zeta}}{\sum_{\zeta=\ell}^{T} \gamma_{\zeta}}\, \phi_h^{j}(W_h^{0}, p_h, p_{h+1}, \cdots, p_T);$$

for $\ell \in \{1,2,\cdots,T\} h \in \{\ell, \ell+1, \cdots, T\}$ and $j \in \{1,2,\cdots,n_h\}$,
where

$W_{\ell} = W_{\ell}^{0}$,
$W_{\ell+1}^{0} = (1+r)(W_{\ell}^{0} - p_{\ell}\phi_{\ell}) + Y_{\ell+1}$,
$W_{\ell+2}^{0} = (1+r)(W_{\ell+1}^{0} - p_{\ell+1}\phi_{\ell+1}) + Y_{\ell+2}$,
$\vdots \qquad \vdots$
$W_h^{0} = (1+r)(W_{h-1}^{0} - p_{h-1}\phi_{h-1}) + Y_h$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5, 2014, pp.263-284.

C3. Inter-temporal Roy's Identity under Stochastic Income

Consider the consumer problem in which the consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \delta_1^{k}\, u^{k}\left(x_k^{1}, x_k^{2}, \cdots, x_k^{n_k}\right) \right\} = E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \delta_1^{k}\, u^{k}\,(x_k) \right\}$$

subject to the budget constraint characterized by the stochastic wealth dynamics

$$W_{k+1} = (1+r)(W_k - p_k x_k) + \theta_{k+1}, \qquad W_1 = W_1^{0},$$

where
θ_k is the random income that the consumer will receive in period k; and θ_k, for $k \in \{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2, \theta_3, \cdots, \theta_T$.

The Roy's Identity of the above stochastic inter-temporal utility problem can be formulated as:

$$\frac{\partial v^{\ell}\,(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial p_{\ell}^{j}} \div \frac{\partial v^{\ell}\,(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial W_{\ell}^{0}}$$

$$\equiv -\phi_{\ell}^{j}(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T), \text{ for } j \in \{1,2,\cdots,n_{\ell}\};$$

$$\frac{\partial v^{\ell}\,(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial p_h^{j}} \div \frac{\partial v^{\ell}\,(W_{\ell}^{0}, p_{\ell}, p_{\ell+1}, \cdots, p_T)}{\partial W_{\ell}^{0}}$$

$$\equiv -\sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_h=1}^{m_h} \lambda_h^{j_h}\, \delta_{\ell+1}^{h} \frac{\partial v^{h}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+1}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p\right)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+1}^{j_{\ell+2}}\cdots\theta_h^{j_h}}}$$

$$\times \frac{\phi_h^{j}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)(1+r)^{-(h-\ell)}}{\sum_{w_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{w_{\ell+1}} \sum_{w_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{w_{\ell+2}} \cdots \sum_{w_h=1}^{m_h} \lambda_h^{w_h} \delta_{\ell+1}^{h} \frac{\partial v^{h}\left(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_{h-1}^{w_{h-1}}}, p\right)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_{h-1}^{w_{h-1}}}}},$$

for $\ell \in \{1,2,\cdots,T\} h \in \{\ell+1, \ell+2, \cdots, T\}$ and $j \in \{1,2,\cdots,n_h\}$,
and $v^{h}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p\right)$ is the short form for $v^{h}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p_h, p_{h+1}, \cdots, p_T\right)$,
where

$W_{\ell} = W_{\ell}^{0}$,
$W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}} = (1+r)[W_{\ell}^{0} - p_{\ell}\phi_{\ell}(W_{\ell}^{0}, p)] + \theta_{\ell+1}^{j_{\ell+1}}$,

$$W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}} = (1+r)\left[W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}} - p_{\ell+1}\phi_{\ell+1}\left(W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}}, p\right)\right] + \theta_{\ell+2}^{j_{\ell+2}},$$

$$\vdots \qquad\qquad \vdots$$

$$W_{T}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T}^{j_{T}}} = (1+r)\left[W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}}} - p_{T-1}\phi_{T-1}\left(W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}}}\right)\right] + \theta_{T}^{j_{T}}. \qquad \blacksquare$$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141.

C4. Inter-temporal Roy's Identity under Stochastic Income and Life-span

Consider the consumer problem in which the consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_{T+1}}\left\{ \sum_{\hat{T}=1}^{T}\gamma_{\hat{T}} \sum_{k=1}^{\hat{T}}\delta_1^k\, u^k(x_k) \right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\left(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\right) + \theta_{k+1}, \qquad W_1 = W_1^0,$$

where

θ_k is the random income that the consumer will receive in period k; and $\hat{T}$ is a random stage that the consumer would live.

The Roy's Identity of the above stochastic inter-temporal utility problem can be formulated as:

$$\frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_\ell^j} \div \frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}$$

$$\equiv -\phi_\ell^j(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T), \text{ for } j \in \{1,2,\cdots,n_\ell\},$$

$$\frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^k} \div \frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}$$

$$\equiv -\sum_{j_{\ell+1}=1}^{j_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\,\delta_{\ell+1}^h\,\frac{\partial v^h\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p\right)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}}\,\phi_h^k(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p)(1+r)^{-(h-\ell)}$$

$$\div\left[\sum_{w_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{w_{\ell+1}}\sum_{w_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{w_{\ell+2}}\cdots\sum_{w_h=1}^{m_h}\lambda_h^{w_h}\,\delta_{l+1}^h\,\frac{\partial v^h\left(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h}}, p\right)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h}}}\right];$$

or in an alternative form:

$$\frac{\partial v^\ell\,(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_h^k}$$

$$\equiv -\frac{\sum_{\zeta=h}^{T}\gamma_\zeta}{\sum_{\zeta=\ell}^{T}\gamma_\zeta}\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\,\delta_{\ell+1}^h\,\frac{\partial v^h\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p\right)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}}\,\phi_h^k\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p\right);$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell+1, \ell+2, \cdots, T\}$ and $k \in \{1,2,\cdots,n_h\}$,

and $v^h\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p\right)$ is the short form for $v^h\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h}}, p_h, p_{h+1}, \cdots, p_T\right)$,

where

$$W_\ell = W_\ell^0,$$

$$W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}} = (1+r)[W_\ell^0 - p_\ell\phi_\ell(W_\ell^0, p)] + \theta_{\ell+1}^{j_{\ell+1}},$$

$$W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}} = (1+r)\left[W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}} - p_{\ell+1}\phi_{\ell+1}\left(W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}}}, p\right)\right] + \theta_{\ell+2}^{j_{\ell+2}},$$

$$\vdots \qquad\qquad \vdots$$

$$W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T}} = (1+r)\left[W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}}} - p_{T-1}\phi_{T-1}\left(W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}}}\right)\right] + \theta_T^{j_T}. \quad \blacksquare$$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5, 2014, pp.263-284.

C5. Inter-temporal Roy's Identity under Stochastic Preferences

Consider the consumer problem in which the consumer's future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\bar{m}_k\}$. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\bar{m}_k\}$ and corresponding probabilities $\{\rho_k^1,\rho_k^2,\cdots,\rho_k^{\bar{m}_k}\}$.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \sum_{\upsilon_k=1}^{\bar{m}_k} \rho_k^{\upsilon_k}\, \delta_1^k u^{k(\upsilon_k)}(x_k) \right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\left(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\right) + Y_{k+1}, \qquad W_1 = W_1^0.$$

The Roy's Identity of the above stochastic inter-temporal utility problem can be formulated as:

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_\ell^j} \div \frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0} \equiv -\phi_\ell^{(\upsilon_\ell)j}(W_\ell^0,p), \text{ for } j \in \{1,2,\cdots,n_\ell\},$$

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_h^k} \div \frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0} \equiv -\sum_{\upsilon_{\ell+1}=1}^{\bar{m}_{\ell+1}} \rho_{\ell+1}^{\upsilon_{\ell+1}} \sum_{\upsilon_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_h=1}^{\bar{m}_h} \rho_h^{\upsilon_h}\, \delta_{\ell+1}^h \frac{\partial v^{h(\upsilon_h)}\left(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p\right)}{\partial W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}} \phi_h^{(\upsilon_h)k}(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)(1+r)^{-(h-\ell)}$$

$$\div \left[\sum_{\varpi_{\ell+1}=1}^{\bar{m}_{\ell+1}} \rho_{\ell+1}^{\varpi_{\ell+1}} \sum_{\varpi_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{\varpi_{\ell+2}} \cdots \sum_{\varpi_h=1}^{\bar{m}_h} \rho_h^{\varpi_h}\, \delta_{\ell+1}^h \frac{\partial v^{h(\varpi_h)}\left(W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p\right)}{\partial W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}} \right];$$

or in an alternative form

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0} \equiv -\sum_{\upsilon_{l+1}=1}^{\bar{m}_{l+1}} \rho_{l+1}^{\upsilon_{l+1}} \sum_{\upsilon_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_{h+1}=1}^{\bar{m}_{h+1}} \rho_h^{\upsilon_h}\, \delta_{\ell+1}^{h+1} \frac{\partial v^{h(\upsilon_h)}\left(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p\right)}{\partial W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}} (1+r)^{h-\ell};$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell+1,\ell+2,\cdots,T\}$, $k \in \{1,2,\cdots,n_h\}$ and $\upsilon_\ell \in \{1,2,\cdots,\bar{m}_\ell\}$,
where

$W_\ell = W_\ell^0$,

$W_{\ell+1}^{\upsilon_\ell} = (1+r)[W_\ell^0 - p_\ell\phi(\upsilon_\ell)_\ell(W_\ell^0,p)] + Y_{\ell+1}$,

$W_{\ell+2}^{\upsilon_\ell\upsilon_{\ell+1}} = (1+r)\left[W_{\ell+1}^{\upsilon_\ell} - p_{\ell+1}\phi_{\ell+1}^{(\upsilon_{\ell+1})}\left(W_{\ell+1}^{\upsilon_\ell},p\right)\right] + Y_{\ell+2}$,

if preference is $u^{\ell+1(\upsilon_{\ell+1})}(x_{\ell+1})$ in period $\ell+1$;

$W_{\ell+3}^{\upsilon_\ell\upsilon_{\ell+1}\upsilon_{\ell+2}} = (1+r)\left[W_{\ell+2}^{\upsilon_\ell\upsilon_{\ell+1}} - p_{\ell+2}\phi_{\ell+2}^{(\upsilon_{\ell+1})}\left(W_{\ell+2}^{\upsilon_\ell\upsilon_{\ell+1}},p\right)\right] + Y_{\ell+3}$,

if preference is $u^{\ell+2(\upsilon_{\ell+2})}(x_{\ell+2})$ in period $\ell+2$;

$\vdots \qquad \vdots$

$W_T^{\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_{T-1}} = (1+r)\left[W_{T-1}^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}} - p_{T-1}\phi_{T-1}^{(\upsilon_{T-1})}\left(W_{T-1}^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}}\right)\right] + Y_T$;

if preference is $u^{T-1(\upsilon_{\ell+2})}(x_{T-1})$ in period $T-1$;

$W_{T+1}^{\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_T} = (1+r)\left[W_T^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}} - p_T\phi_T^{(\upsilon\ \ )}\left(W_T^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}}\right)\right] + Y_{T+1} = 0.$ $\blacksquare$

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

C6. Inter-temporal Roy's Identity under Stochastic Life-span and Preferences

Consider the consumer problem in which the consumer's future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\bar{m}_k\}$ if he survives in period k. We use $\tilde{\upsilon}_k$to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\bar{m}_k\}$ and corresponding probabilities $\{\rho_k^1,\rho_k^2,\cdots,\rho_k^{\bar{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{\hat{T}=1}^{T}\gamma_{\hat{T}}\sum_{k=1}^{\hat{T}} \sum_{\upsilon_k=1}^{\bar{m}_k}\rho_k^{\upsilon_k}\,\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$
$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1)+\sum_{\hat{T}=2}^{T}\gamma_{\hat{T}}\sum_{k=2}^{\hat{T}} \sum_{\upsilon_k=1}^{\bar{m}_k}\rho_k^{\upsilon_k}\,\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1}=W_k-\sum_{h=1}^{n_k}p_k^h x_k^h + r\left(W_k-\sum_{h=1}^{n_k}p_k^h x_k^h\right)+Y_{k+1}, \qquad W_1=W_1^0.$$

The Roy's Identity of the above stochastic inter-temporal utility problem can be formulated as:

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_\ell^j}\div\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0}\equiv-\phi_\ell^{(\upsilon_\ell)j}(W_\ell^0,p), \text{ for } j\in\{1,2,\cdots,n_\ell\},$$
$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_h^k}\div\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0}\equiv-\sum_{\upsilon_{\ell+1}=1}^{\bar{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\bar{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots$$
$$\cdots\sum_{\upsilon_h=1}^{\bar{m}_h}\rho_h^{\upsilon_h}\,\delta_{\ell+1}^h\frac{\partial v^{h(\upsilon_h)}\left(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p\right)}{\partial W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}\phi_h^{(\upsilon_h)k}(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)(1+r)^{-(h-\ell)}$$
$$\div\left[\sum_{\varpi_{\ell+1}=1}^{\bar{m}_{\ell+1}}\rho_{\ell+1}^{\varpi_{\ell+1}}\sum_{\varpi_{\ell+2}=1}^{\bar{m}_{\ell+2}}\rho_{\ell+2}^{\varpi_{\ell+2}}\cdots\sum_{\varpi_h=1}^{\bar{m}_h}\rho_h^{\varpi_h}\,\delta_{\ell+1}^h\frac{\partial v^{h(\varpi_h)}\left(W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p\right)}{\partial W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}}\right];$$

or in an alternative form

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0}\equiv-\frac{\sum_{\zeta=h}^{T}\gamma_\zeta}{\sum_{\zeta=\ell}^{T}\gamma_\zeta}\sum_{\upsilon_{l+1}=1}^{\bar{m}_{l+1}}\rho_{l+1}^{\upsilon_{l+1}}\sum_{\upsilon_{\ell+2}=1}^{\bar{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots$$
$$\cdots\sum_{\upsilon_{h+1}=1}^{\bar{m}_{h+1}}\rho_h^{\upsilon_h}\,\delta_{\ell+1}^{h+1}\frac{\partial v^{h(\upsilon_h)}\left(W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p\right)}{\partial W_h^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}(1+r)^{h-\ell};$$

for $\ell\in\{1,2,\cdots,T\}$, $h\in\{\ell+1,\ell+2,\cdots,T\}$, $k\in\{1,2,\cdots,n_h\}$ and $\upsilon_\ell\in\{1,2,\cdots,\bar{m}_\ell\}$,

where

$W_\ell=W_\ell^0$,

$W_{\ell+1}^{\upsilon_\ell}=(1+r)[W_\ell^0-p_\ell\phi(\upsilon_\ell)_\ell(W_\ell^0,p)]+Y_{\ell+1}$,

$W_{\ell+2}^{\upsilon_\ell\upsilon_{\ell+1}}=(1+r)\left[W_{\ell+1}^{\upsilon_\ell}-p_{\ell+1}\phi_{\ell+1}^{(\upsilon_{\ell+1})}(W_{\ell+1}^{\upsilon_\ell},p)\right]+Y_{\ell+2}$,

if preference is $u^{\ell+1(\upsilon_{\ell+1})}(x_{\ell+1})$ in period $\ell+1$;

$W_{\ell+3}^{\upsilon_\ell\upsilon_{\ell+1}\upsilon_{\ell+2}}=(1+r)\left[W_{\ell+2}^{\upsilon_\ell\upsilon_{\ell+1}}-p_{\ell+2}\phi_{\ell+2}^{(\upsilon_{\ell+1})}(W_{\ell+2}^{\upsilon_\ell\upsilon_{\ell+1}},p)\right]+Y_{\ell+3}$,

if preference is $u^{\ell+2(\upsilon_{\ell+2})}(x_{\ell+2})$ in period $\ell+2$;

⋮ ⋮

$W_T^{\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_{T-1}}=(1+r)\left[W_{T-1}^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}}-p_{T-1}\phi_{T-1}^{(\upsilon_{T-1})}(W_{T-1}^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}})\right]+Y_T$;

if preference is $u^{T-1(\upsilon_{\ell+2})}(x_{T-1})$ in period $T-1$;

$W_{T+1}^{\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_T}=(1+r)\left[W_T^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}}-p_T\phi_T^{(\upsilon\ \ )}(W_T^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}})\right]+Y_{T+1}=0.$ ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

C7. Inter-temporal Roy's Identity under Stochastic Income and Preferences

Consider the consumer problem in which the consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \sum_{\upsilon_k=1}^{\bar{m}_k}\rho_k^{\upsilon_k}\,\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$
$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1)+\sum_{k=2}^{T} \sum_{\upsilon_k=1}^{\bar{m}_k}\rho_k^{\upsilon_k}\,\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamic

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\big(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\big) + \theta_{k+1}, \qquad W_1 = W_1^0,$$

where θ_k is the random income that the consumer will receive in period k.

The Roy's Identity of the above stochastic inter-temporal utility problem can be formulated as:

$$\frac{\partial v^{\ell(v_\ell)}(W_\ell^0,p)}{\partial p_\ell^j} \div \frac{\partial v^{\ell(v_\ell)}(W_\ell^0,p)}{\partial W_\ell^0} \equiv -\phi_\ell^{(v_\ell)j}(W_\ell^0,p), \text{ for } j \in \{1,2,\cdots,n_\ell\};$$

$$\frac{\partial v^{\ell(v_\ell)}(W_\ell^0,p)}{\partial p_h^k} \div \frac{\partial v^{\ell(v_\ell)}(W_\ell^0,p)}{\partial W_\ell^0} \equiv -\sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_h=1}^{m_h} \lambda_h^{j_h} \sum_{v_{\ell+1}=1}^{\bar m_{\ell+1}} \rho_{\ell+1}^{v_{\ell+1}} \sum_{v_{\ell+2}=1}^{\bar m_{\ell+2}} \rho_{\ell+2}^{v_{\ell+2}} \cdots$$

$$\cdots \sum_{v_h=1}^{\bar m_h} \rho_h^{v_h}\, \delta_{l+1}^h \frac{\partial v^{h(v_h)}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};v_\ell v_{\ell+1}\cdots v_{h-1}},p\right)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};v_\ell v_{\ell+1}\cdots v_{h-1}}}$$

$$\times \phi_h^{(v_h)k}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};v_\ell v_{\ell+1}\cdots v_{h-1}},p)(1+r)^{-(h-\ell)}$$

$$\div \Bigg[\sum_{w_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{w_{\ell+1}} \sum_{w_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{w_{\ell+2}} \cdots \sum_{w_h=1}^{m_h} \lambda_h^{w_h} \sum_{\varpi_{\ell+1}=1}^{\bar m_{\ell+1}} \rho_{\ell+1}^{\varpi_{\ell+1}} \sum_{\varpi_{\ell+2}=1}^{\bar m_{\ell+2}} \rho_{\ell+2}^{\varpi_{\ell+2}} \cdots \sum_{\varpi_h=1}^{\bar m_h} \rho_h^{\varpi_h}$$

$$\times\, \delta_{l+1}^h \frac{\partial v^{h(\varpi_h)}\left(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell \varpi_{\ell+1}\cdots\varpi_{h-1}},p\right)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}} \Bigg];$$

or in an alternative form:

$$\frac{\partial v^{\ell(v_\ell)}(W_\ell^0,p)}{\partial p_h^k} \equiv -\sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_h=1}^{m_h} \lambda_h^{j_h} \sum_{v_{\ell+1}=1}^{\bar m_{\ell+1}} \rho_{\ell+1}^{v_{\ell+1}} \sum_{v_{\ell+2}=1}^{\bar m_{\ell+2}} \rho_{\ell+2}^{v_{\ell+2}} \cdots \sum_{v_h=1}^{\bar m_h} \rho_h^{v_h}$$

$$\delta_{\ell+1}^h \frac{\partial v^{h(v_h)}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};v_\ell v_{\ell+1}\cdots v_{h-1}},p\right)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};v_\ell v_{\ell+1}\cdots v_{h-1}}} \phi_h^{(v_h)k}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};v_\ell v_{\ell+1}\cdots v_{h-1}},p\right);$$

for $\ell \in \{1,2,\cdots,T\}$, $h \in \{\ell+1,\ell+2,\cdots,T\}$, $k \in \{1,2,\cdots,n_h\}$ and $v_\ell \in \{1,2,\cdots,\bar m_\ell\}$,

where

$W_\ell = W_\ell^0$,

$$W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};v_\ell} = (1+r)[W_\ell^0 - p_\ell \phi(v_\ell)_\ell(W_\ell^0,p)] + \theta_{\ell+1}^{j_{\ell+1}},$$

$$W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};v_\ell v_{\ell+1}} = (1+r)\left[W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};v_\ell} - p_{\ell+1}\phi_{\ell+1}^{(v_{\ell+1})}\left(W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};v_\ell},p\right)\right] + \theta_{\ell+2}^{j_{\ell+2}},$$

if preference is $u^{\ell+1(v_{\ell+1})}(x_{\ell+1})$ in period $\ell+1$;

$$W_{\ell+3}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\theta_{\ell+3}^{j_{\ell+3}};v_\ell v_{\ell+1} v_{\ell+2}} = (1+r)\left[W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};v_\ell v_{\ell+1}} - p_{\ell+2}\phi_{\ell+2}^{(v_{\ell+1})}\left(W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};v_\ell v_{\ell+1}},p\right)\right] + \theta_{\ell+3}^{j_{\ell+3}},$$

if preference is $u^{\ell+2(v_{\ell+2})}(x_{\ell+2})$ in period $\ell+2$;

$\vdots \qquad\qquad \vdots$

$$W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T};v_\ell,v_{\ell+1}\cdots v_{T-1}}$$

$$= (1+r)\left[W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}};v_\ell v_{\ell+1}\cdots v_{T-2}} - p_{T-1}\phi_{T-1}^{(v_{T-1})}\left(W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}};v_\ell v_{\ell+1}\cdots v_{T-2}}\right)\right] + \theta_T^{j_T};$$

if preference is $u^{T-1(v_{\ell+2})}(x_{T-1})$ in period $T-1$;

$$W_{T+1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T+1}^{j_{T+1}};v_\ell,v_{\ell+1}\cdots v_T}$$

$$= (1+r)\left[W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T};v_\ell v_{\ell+1}\cdots v_{T-1}} - p_T\phi_T^{(v\ \ )}\left(W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T};v_\ell v_{\ell+1}\cdots v_{T-1}}\right)\right] + \theta_{T+1}^{j_{T+1}} = 0.$$

■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

C8. Inter-temporal Roy's Identity under Stochastic Income, Life-span and Preferences

Consider the consumer problem in which the consumer's future income, preferences and life span are not known with certainty. The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{\hat{T}=1}^{T}\gamma_{\hat{T}}\sum_{k=1}^{\hat{T}} \sum_{\upsilon_k=1}^{\bar{m}_k}\rho_k^{\upsilon_k}\,\delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1)+\sum_{\hat{T}=2}^{T}\gamma_{\hat{T}}\sum_{k=2}^{\hat{T}} \sum_{\upsilon_k=1}^{\bar{m}_k}\rho_k^{\upsilon_k}\,\delta_1^k u^{k(\upsilon_k)}(x_k) \right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1}=W_k-\sum_{h=1}^{n_k}p_k^h x_k^h + r\left(W_k-\sum_{h=1}^{n_k}p_k^h x_k^h\right)+\theta_{k+1}, \qquad W_1=W_1^0.$$

The Roy's Identity of the above stochastic inter-temporal utility problem can be formulated as:

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_\ell^j}\div\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0}\equiv-\phi_\ell^{(\upsilon_\ell)j}(W_\ell^0,p), \text{ for } j\in\{1,2,\cdots,n_\ell\};$$

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_h^k}\div\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial W_\ell^0}\equiv-\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\sum_{\upsilon_{\ell+1}=1}^{\bar{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\bar{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots$$

$$\cdots\sum_{\upsilon_h=1}^{\bar{m}_h}\rho_h^{\upsilon_h}\frac{\partial v^{h(\upsilon_h)}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p\right)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}\phi_h^{(\upsilon_h)k}(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p)(1+r)^{-(h-\ell)}$$

$$\div\Bigg[\sum_{w_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{w_{\ell+1}}\sum_{w_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{w_{\ell+2}}\cdots\sum_{w_h=1}^{m_h}\lambda_h^{w_h}\sum_{\varpi_{\ell+1}=1}^{\bar{m}_{\ell+1}}\rho_{\ell+1}^{\varpi_{\ell+1}}\sum_{\varpi_{\ell+2}=1}^{\bar{m}_{\ell+2}}\rho_{\ell+2}^{\varpi_{\ell+2}}\cdots\sum_{\varpi_h=1}^{\bar{m}_h}\rho_h^{\varpi_h}$$

$$\times\,\delta_{\ell+1}^h\frac{\partial v^{h(\varpi_h)}\left(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p\right)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}}\Bigg];$$

or in an alternative form:

$$\frac{\partial v^{\ell(\upsilon_\ell)}(W_\ell^0,p)}{\partial p_h^k}\equiv-\frac{\sum_{\zeta=h}^{T}\gamma_\zeta}{\sum_{\zeta=\ell}^{T}\gamma_\zeta}\sum_{j_{\ell+1}=1}^{m_{\ell+1}}\lambda_{\ell+1}^{j_{\ell+1}}\sum_{j_{\ell+2}=1}^{m_{\ell+2}}\lambda_{\ell+2}^{j_{\ell+2}}\cdots\sum_{j_h=1}^{m_h}\lambda_h^{j_h}\sum_{\upsilon_{\ell+1}=1}^{\bar{m}_{\ell+1}}\rho_{\ell+1}^{\upsilon_{\ell+1}}\sum_{\upsilon_{\ell+2}=1}^{\bar{m}_{\ell+2}}\rho_{\ell+2}^{\upsilon_{\ell+2}}\cdots\sum_{\upsilon_h=1}^{\bar{m}_h}\rho_h^{\upsilon_h}$$

$$\delta_{\ell+1}^h\frac{\partial v^{h(\upsilon_h)}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p\right)}{\partial W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}}}\phi_h^{(\upsilon_h)k}\left(W_h^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_h^{j_h};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{h-1}},p\right);$$

for $\ell\in\{1,2,\cdots,T\}$, $h\in\{\ell+1,\ell+2,\cdots,T\}$, $k\in\{1,2,\cdots,n_h\}$ and $\upsilon_\ell\in\{1,2,\cdots,\bar{m}_\ell\}$,
where

$W_\ell=W_\ell^0$,

$W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell}=(1+r)[W_\ell^0-p_\ell\phi(\upsilon_\ell)_\ell(W_\ell^0,p)]+\theta_{\ell+1}^{j_{\ell+1}}$,

$W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}}=(1+r)\left[W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell}-p_{\ell+1}\phi_{\ell+1}^{(\upsilon_{\ell+1})}\left(W_{\ell+1}^{\theta_{\ell+1}^{j_{\ell+1}};\upsilon_\ell},p\right)\right]+\theta_{\ell+2}^{j_{\ell+2}}$,

if preference is $u^{\ell+1(\upsilon_{\ell+1})}(x_{\ell+1})$ in period $\ell+1$;

$W_{\ell+3}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\theta_{\ell+3}^{j_{\ell+3}};\upsilon_\ell\upsilon_{\ell+1}\upsilon_{\ell+2}}=(1+r)\left[W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}}-p_{\ell+2}\phi_{\ell+2}^{(\upsilon_{\ell+1})}\left(W_{\ell+2}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}};\upsilon_\ell\upsilon_{\ell+1}},p\right)\right]+$
$\theta_{\ell+3}^{j_{\ell+3}}$,

if preference is $u^{\ell+2(\upsilon_{\ell+2})}(x_{\ell+2})$ in period $\ell+2$;

$\vdots \qquad\qquad \vdots$

$W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}}$

$=(1+r)\left[W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}}-\right.$

$\left.p_{T-1}\phi_{T-1}^{(\upsilon_{T-1})}\left(W_{T-1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T-1}^{j_{T-1}};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-2}}\right)\right]+\theta_T^{j_T}$;

if preference is $u^{T-1(\upsilon_{\ell+2})}(x_{T-1})$ in period $T-1$;

$$W_{T+1}^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_{T+1}^{j_{T+1}};\upsilon_\ell,\upsilon_{\ell+1}\cdots\upsilon_T}$$
$$= (1+r)\left[W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}} - p_T\phi_T^{(\upsilon\ \ )}\left(W_T^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_T^{j_T};\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{T-1}}\right)\right] + \theta_{T+1}^{j_{T+1}} = 0.$$

■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

C9. Dynamic Slutsky Equation

Consider the consumer problem in which the consumer maximizes his inter-temporal utility

$$u^1\left(x_1^1, x_1^2, \cdots, x_1^{n_1}\right) + \sum_{k=2}^T \delta_2^k\, u^k\left(x_k^1, x_k^2, \cdots, x_k^{n_k}\right)$$
$$= u^1(x_1) + \sum_{k=2}^T \delta_2^k\, u^k(x_k) = \sum_{k=1}^T \delta_1^k\, u^k(x_k)$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{\eta_k} p_k^h x_k^h + r(W_k - \sum_{h=1}^{\eta_k} p_k^h x_k^h) + Y_{k+1}, \quad W_1 = W_1^0,$$

where
$x_k = \left(x_k^1, x_k^2, \cdots, x_k^{n_k}\right)$ is the vector of quantities of goods consumed in period k, $p_k = \left(p_k^1, p_k^2, \cdots, p_k^{n_k}\right)$ is price vector, r is the interest rate, and Y_k is the income that the consumer will receive in period k.

The dynamic Slutsky Equation of the above inter-temporal utility problem can be formulated as:

$$\frac{\partial\phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_k^j} = \frac{\partial\psi_\ell^h\left(\hat{\upsilon}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T\right)}{\partial p_k^j}$$
$$-\frac{\partial\varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_l^0}\phi_k^j(W_k^0, p_k, p_{k+1}, \cdots, p_T)(1+r)^{-(k-\ell)},$$

for $j \in \{1,2,\cdots,n_k\}$ and $k \in \{\ell, \ell+1, \cdots, T\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141.

C10. Dynamic Slutsky Equation under Stochastic Life-span

Consider the consumer problem in C1 with the consumer's life-span involves $\hat{T}$ periods where $\hat{T}$ is a random variable with range $\{1,2,\cdots,T\}$ and corresponding probabilities $\{\gamma_1, \gamma_2, \cdots, \gamma_T\}$. Conditional upon the reaching of period τ, the probability of the consumer's life-span would last up to periods $\tau, \tau+1, \cdots, T$ becomes respectively

$$\frac{\gamma_\tau}{\sum_{\zeta=\tau}^T \gamma_\zeta}, \frac{\gamma_{\tau+1}}{\sum_{\zeta=\tau}^T \gamma_\zeta}, \cdots, \frac{\gamma_T}{\sum_{\zeta=\tau}^T \gamma_\zeta}.$$

The consumer maximizes his expected inter-temporal utility

$$\sum_{\hat{T}=1}^T \gamma_{\hat{T}} \sum_{k=1}^{\hat{T}} \delta_1^k\, u^k(x_k),$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\left(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\right) + Y_{k+1}, \qquad W_1 = W_1^0.$$

The dynamic Slutsky equation of the above stochastic life-span is formulated as:

$$\frac{\partial\phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_k^j} = \frac{\partial\psi_\ell^h\left(\hat{\upsilon}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T\right)}{\partial p_k^j}$$
$$-\frac{\partial\phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0}(1+r)^{-(k-\ell)}\phi_k^j(W_k^0, p_k, p_{k+1}, \cdots, p_T),$$

for $j \in \{1,2,\cdots,n_k\}$ and $k \in \{\ell, \ell+1, \cdots, T\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5, 2014, pp.263-284.

C11. Dynamic Slutsky Equation under Stochastic Income

Consider the consumer problem in which the consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3\cdots,\theta_T}\left\{ \quad \sum_{k=1}^{T} \delta_1^k\, u^k\left(x_k^1, x_k^2, \cdots, x_k^{n_k}\right) \quad \right\} = E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \quad \sum_{k=1}^{T} \delta_1^k\, u^k\,(x_k) \quad \right\}$$

subject to the budget constraint characterized by the stochastic wealth dynamics

$$W_{k+1} = (1+r)(W_k - p_k x_k) + \theta_{k+1}, \qquad W_1 = W_1^0,$$

where
θ_k is the random income that the consumer will receive in period k; and θ_k, for $k \in \{2,\cdots,T\}$, is a set of statistically independent random variables, and $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operation with respect to the statistics of $\theta_2,\theta_3,\cdots,\theta_T$.

The dynamic Slutsky equation of the above stochastic inter-temporal utility problem is formulated as:

$$\frac{\partial \phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_\ell^j} = \frac{\partial \psi_\ell^h\left(\hat{v}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T\right)}{\partial p_\ell^j}$$
$$- \frac{\partial \varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_l^0}\, \phi_\ell^j(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T),$$

for $j \in \{1,2,\cdots,n_\ell\}$,
and

$$\frac{\partial \phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_k^j} = \frac{\partial \psi_\ell^h\left(\hat{v}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T\right)}{\partial p_k^j}$$
$$- \frac{\partial \phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0} \sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_k=1}^{m_k} \lambda_k^{j_k}$$
$$\times \frac{\delta_{l+1}^k \dfrac{\partial v^k\left(W_k^{\theta_{l+1}^{j_{l+1}}\theta_{l+2}^{j_{l+2}}\cdots\theta_k^{j_k}}, p\right)}{\partial W_k^{\theta_{l+1}^{j_{l+1}}\theta_{l+2}^{j_{l+2}}\cdots\theta_k^{j_k}}}}{\sum_{w_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{w_{\ell+1}} \sum_{w_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{w_{\ell+2}} \cdots \sum_{w_k=1}^{m_k} \lambda_k^{w_k} \delta_{l+1}^k \dfrac{\partial v^k\left(W_k^{\theta_{l+1}^{w_{l+1}}\theta_{l+2}^{w_{l+2}}\cdots\theta_k^{w_k}}, p\right)}{\partial W_k^{\theta_{l+1}^{w_{l+1}}\theta_{l+2}^{w_{l+2}}\cdots\theta_k^{w_k}}}}\, \phi_k^j\left(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k}}, p\right)(1 +$$
$$r)^{-(k-\ell)},$$

for $\ell \in \{1,2,\cdots,T-1\}, k \in \{k+1, k+2, \cdots, T\}$ and $j \in \{1,2,\cdots,n_k\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Optimal Consumption under an Uncertain Inter-temporal Budget: Stochastic Dynamic Slutsky Equations, Vestnik St Petersburg University: Mathematics (Springer), Vol. 10, 2013, pp.121-141.

C12. Dynamic Slutsky Equation under Stochastic Income and Life-span

Consider the consumer problem in which the consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_{T+1}}\left\{ \quad \sum_{\hat{T}=1}^{T} \gamma_{\hat{T}} \sum_{k=1}^{\hat{T}} \delta_1^k\, u^k\,(x_k) \quad \right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\left(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\right) + \theta_{k+1}, \qquad W_1 = W_1^0,$$

where

θ_k is the random income that the consumer will receive in period k; and $\hat{T}$ is a random stage that the consumer would live.

The dynamic Slutsky equation of the above stochastic inter-temporal utility problem is formulated as:

$$\frac{\partial \phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_\ell^j} = \frac{\partial \psi_\ell^h\left(\hat{v}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T\right)}{\partial p_\ell^j}$$

$$-\frac{\partial \varphi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0} \phi_\ell^j(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T),$$

for $j \in \{1,2,\cdots,n_\ell\}$,

and

$$\frac{\partial \phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial p_k^j} = \frac{\partial \psi_\ell^h\left(\hat{v}_\ell^{W_\ell^0}, p_\ell, p_{\ell+1}, \cdots, p_T\right)}{\partial p_k^j}$$

$$-\frac{\partial \phi_\ell^h(W_\ell^0, p_\ell, p_{\ell+1}, \cdots, p_T)}{\partial W_\ell^0} \sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+2}} \cdots \sum_{j_k=1}^{m_k} \lambda_k^{j_k}$$

$$\times \frac{\delta_{l+1}^k \dfrac{\partial v^k\left(W_k^{\theta_{l+1}^{j_{l+1}}\theta_{l+2}^{j_{l+2}}\cdots\theta_k^{j_k}}, p\right)}{\partial W_k^{\theta_{l+1}^{j_{l+1}}\theta_{l+2}^{j_{l+2}}\cdots\theta_k^{j_k}}}}{\sum_{w_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{w_{\ell+1}} \sum_{w_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{w_{\ell+2}} \cdots \sum_{w_k=1}^{m_k} \lambda_k^{w_k} \delta_{l+1}^k \dfrac{\partial v^k\left(W_k^{\theta_{l+1}^{w_{l+1}}\theta_{l+2}^{w_{l+2}}\cdots\theta_k^{w_k}}, p\right)}{\partial W_k^{\theta_{l+1}^{w_{l+1}}\theta_{l+2}^{w_{l+2}}\cdots\theta_k^{w_k}}}}$$

$$\phi_\ell^j(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k}}, p)(1+r)^{-(k-\ell)},$$

for $\ell \in \{1,2,\cdots,T-1\}$, $k \in \{k+1, k+2, \cdots, T\}$ and $j \in \{1,2,\cdots,n_k\}$.

■ ***References:*** *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Optimal Consumption under Uncertainties: Random Horizon Stochastic Dynamic Roy's Identity and Slutsky Equation, Applied Mathematics, Vol.5, 2014, pp.263-284.

C13. Dynamic Slutsky Equation under Stochastic Preferences

Consider the consumer problem in which the consumer's future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\bar{m}_k\}$. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\bar{m}_k\}$ and corresponding probabilities $\{\rho_k^1, \rho_k^2, \cdots, \rho_k^{\bar{m}_k}\}$.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_{T+1}}\left\{ \sum_{k=1}^{T} \sum_{\upsilon_k=1}^{\bar{m}_k} \rho_k^{\upsilon_k} \delta_1^k u^{k(\upsilon_k)}(x_k) \right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\left(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\right) + Y_{k+1}, \qquad W_1 = W_1^0.$$

The dynamic Slutsky equation of the above stochastic inter-temporal utility problem is formulated as:

$$\frac{\partial \phi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial p_\ell^{i_\ell}} = \frac{\partial \psi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial p_\ell^{i_\ell}} - \frac{\partial \varphi_\ell^{(\upsilon_l)h}(W_\ell^0, p)}{\partial W_\ell^0} \phi_\ell^{i_\ell}(W_\ell^0, p),$$

$$\frac{\partial \phi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial p_k^{i_k}} = \frac{\partial \psi_\ell^{(\upsilon_\ell)h}(W_\ell^0, p)}{\partial p_k^{i_k}} - \frac{\partial \varphi_\ell^{(\upsilon_l)h}(W_\ell^0, p)}{\partial W_\ell^0} \sum_{\upsilon_{\ell+1}=1}^{\bar{m}_{\ell+1}} \rho_{\ell+1}^{\upsilon_{\ell+1}} \sum_{\upsilon_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_k=1}^{\bar{m}_k} \rho_k^{\upsilon_k} \frac{\partial v^{(\upsilon_h)k}\left(W_k^{\upsilon_\ell \upsilon_{\ell+1}\cdots\upsilon_{k-1}}, p\right)}{\partial W_k^{\upsilon_\ell \upsilon_{\ell+1}\cdots\upsilon_{k-1}}} \phi_k^{i_k}(W_k^{\upsilon_\ell \upsilon_{\ell+1}\cdots\upsilon_{k-1}}, p)(1+r)^{-(\ell-k)}$$

$$\div \left[\sum_{\varpi_{l+1}=1}^{\bar{m}_{l+1}} \rho_{l+1}^{l+1} \sum_{\varpi_{l+2}=1}^{\bar{m}_{l+2}} \rho_{l+2}^{l+2} \cdots \sum_{\varpi_h=1}^{\bar{m}_h} \rho_h^{\varpi_h} \delta_{\ell+1}^h \frac{\partial v^{h(\varpi_h)}\left(W_h^{\varpi_\ell \varpi_{\ell+1}\cdots\varpi_{h-1}}, p\right)}{\partial W_h^{\varpi_\ell \varpi_{\ell+1}\cdots\varpi_{h-1}}} \right],$$

for $\ell \in \{1,2,\cdots,T\}$, $k \in \{\ell+1, \ell+2, \cdots, T\}$, $i_k \in \{1,2,\cdots,n_k\}$, $h, i_\ell \in \{1,2,\cdots,n_\ell\}$ and
$\upsilon_\ell \in \{1,2,\cdots,\bar{m}_\ell\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

C14. Dynamic Slutsky Equation under Stochastic Life-span and Preferences

Consider the consumer problem in which the consumer's future preferences are not known with certainty. In particular, his utility function in period $k \in \{2,3,\cdots,T\}$ is known to be $u^{k(\upsilon_k)}(x_k)$ with probability $\rho_k^{\upsilon_k}$ for $\upsilon_k \in \{1,2,\cdots,\bar{m}_k\}$ if he survives in period k. We use $\tilde{\upsilon}_k$ to denote the random variable with range $\upsilon_k \in \{1,2,\cdots,\bar{m}_k\}$ and corresponding probabilities $\{\rho_k^1, \rho_k^2, \cdots, \rho_k^{\bar{m}_k}\}$. The discount factor is embodied in the utility function.

The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{\hat{T}=1}^{T} \gamma_{\hat{T}} \sum_{k=1}^{\hat{T}} \sum_{\upsilon_k=1}^{\bar{m}_k} \rho_k^{\upsilon_k}\, \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{\hat{T}=2}^{T} \gamma_{\hat{T}} \sum_{k=2}^{\hat{T}} \sum_{\upsilon_k=1}^{\bar{m}_k} \rho_k^{\upsilon_k}\, \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\left(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\right) + Y_{k+1}, \qquad W_1 = W_1^0.$$

The dynamic Slutsky equation of the above stochastic inter-temporal utility problem is formulated as:

$$\frac{\partial \phi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} = \frac{\partial \psi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} - \frac{\partial \varphi_\ell^{(\upsilon_l)h}(W_\ell^0,p)}{\partial W_\ell^0} \phi_\ell^{i_\ell}(W_\ell^0,p),$$

$$\frac{\partial \phi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} = \frac{\partial \psi_\ell^{(\upsilon_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} - \frac{\partial \varphi_\ell^{(\upsilon_l)h}(W_\ell^0,p)}{\partial W_\ell^0} \sum_{\upsilon_{\ell+1}=1}^{\bar{m}_{\ell+1}} \rho_{\ell+1}^{\upsilon_{\ell+1}} \sum_{\upsilon_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{\upsilon_{\ell+2}} \cdots$$

$$\cdots \sum_{\upsilon_k=1}^{\bar{m}_k} \rho_k^{\upsilon_k}\, \delta_{\ell+1}^k \frac{\partial v^{(\upsilon_k)k}\left(W_k^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}},p\right)}{\partial W_k^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}}} \phi_k^{i_k}(W_k^{\upsilon_\ell\upsilon_{\ell+1}\cdots\upsilon_{k-1}},p)(1+r)^{-(\ell-k)}$$

$$\div \left[\sum_{\varpi_{l+1}=1}^{\bar{m}_{l+1}} \rho_{l+1}^{\varpi_{l+1}} \sum_{\varpi_{l+2}=1}^{\bar{m}_{l+2}} \rho_{l+2}^{\varpi_{l+2}} \cdots \sum_{\varpi_h=1}^{\bar{m}_h} \rho_h^{\varpi_h}\, \delta_{\ell+1}^h \frac{\partial v^{h(\varpi_h)}\left(W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p\right)}{\partial W_h^{\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}} \right],$$

for $\ell \in \{1,2,\cdots,T\}$, $k \in \{\ell+1, \ell+2, \cdots, T\}$, $i_k \in \{1,2,\cdots,n_k\}$, $h, i_\ell \in \{1,2,\cdots,n_\ell\}$ and
$\upsilon_\ell \in \{1,2,\cdots,\bar{m}_\ell\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

C15. Dynamic Slutsky Equation under Stochastic Income and Preferences

Consider the consumer problem in which the consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} \sum_{\upsilon_k=1}^{\bar{m}_k} \rho_k^{\upsilon_k}\, \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{k=2}^{T} \sum_{\upsilon_k=1}^{\bar{m}_k} \rho_k^{\upsilon_k}\, \delta_1^k u^{k(\upsilon_k)}(x_k) \right\}$$

subject to the budget constraint characterized by the wealth dynamic

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\left(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\right) + \theta_{k+1}, \qquad W_1 = W_1^0,$$

where θ_k is the random income that the consumer will receive in period k.

The dynamic Slutsky equation of the above stochastic inter-temporal utility is formulated as:

$$\frac{\partial\phi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} = \frac{\partial\psi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} - \frac{\partial\varphi_\ell^{(v_l)h}(W_\ell^0,p)}{\partial W_\ell^0}\phi_\ell^{i_\ell}(W_\ell^0,p),$$

$$\frac{\partial\phi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} = \frac{\partial\psi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} - \frac{\partial\phi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}$$

$$\times \sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+1}} \cdots \sum_{j_\ell=1}^{m_\ell} \lambda_k^{j_k} \sum_{v_{\ell+1}=1}^{\bar{m}_{\ell+1}} \rho_{\ell+1}^{v_{\ell+1}} \sum_{v_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{v_{\ell+2}} \cdots$$

$$\cdots \sum_{v_k=1}^{\bar{m}_k} \rho_k^{v_k}\,\delta_{\ell+1}^k \frac{\partial v^{(v_k)k}\left(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};v_\ell v_{\ell+1}\cdots v_{k-1}},p\right)}{\partial W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};v_\ell v_{\ell+1}\cdots v_{k-1}}} \phi_k^{i_k}(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};v_\ell v_{\ell+1}\cdots v_{k-1}},p)(1+r)^{-(\ell-k)}$$

$$\div \Bigg[\sum_{w_{l+1}=1}^{m_{l+1}} \lambda_{l+1}^{w_{l+1}} \sum_{w_{l+2}=1}^{m_{l+2}} \lambda_{l+2}^{w_{l+2}} \cdots \sum_{w_k=1}^{m_k} \lambda_k^{w_k} \sum_{\varpi_{\ell+1}=1}^{\bar{m}_{\ell+1}} \rho_{\ell+1}^{\varpi_{\ell+1}} \sum_{\varpi_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{\varpi_{\ell+2}} \cdots$$

$$\cdots \sum_{\varpi_h=1}^{\bar{m}_h} \rho_h^{\varpi_h}\,\delta_{\ell+1}^h \frac{\partial v^{h(\varpi_h)}\left(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p\right)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}} \Bigg],$$

for $\ell \in \{1,2,\cdots,T\}$, $k \in \{\ell+1,\ell+2,\cdots,T\}$, $i_k \in \{1,2,\cdots,n_k\}$, $h, i_\ell \in \{1,2,\cdots,n_\ell\}$ and $v_\ell \in \{1,2,\cdots,\bar{m}_\ell\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*

D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

C16. Dynamic Slutsky Equation under Stochastic Income, Life-span and Preferences

Consider the consumer problem in which the consumer's future income, preferences and life span are not known with certainty. The consumer maximizes his expected inter-temporal utility

$$E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ \sum_{\hat{T}=1}^{T}\gamma_{\hat{T}}\sum_{k=1}^{\hat{T}} \sum_{v_k=1}^{\bar{m}_k}\rho_k^{v_k}\,\delta_1^k u^{k(v_k)}(x_k) \right\}$$
$$= E_{\theta_2,\theta_3,\cdots,\theta_T}\left\{ u^{1(1)}(x_1) + \sum_{\hat{T}=2}^{T}\gamma_{\hat{T}}\sum_{k=2}^{\hat{T}} \sum_{v_k=1}^{\bar{m}_k}\rho_k^{v_k}\,\delta_1^k u^{k(v_k)}(x_k) \right\},$$

subject to the budget constraint characterized by the wealth dynamics

$$W_{k+1} = W_k - \sum_{h=1}^{n_k} p_k^h x_k^h + r\big(W_k - \sum_{h=1}^{n_k} p_k^h x_k^h\big) + \theta_{k+1}, \qquad W_1 = W_1^0.$$

The dynamic Slutsky equation of the above stochastic inter-temporal utility problem is formulated as:

$$\frac{\partial\phi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} = \frac{\partial\psi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial p_\ell^{i_\ell}} - \frac{\partial\varphi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}\phi_\ell^{i_\ell}(W_\ell^0,p),$$

$$\frac{\partial\phi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} = \frac{\partial\psi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial p_k^{i_k}} - \frac{\partial\phi_\ell^{(v_\ell)h}(W_\ell^0,p)}{\partial W_\ell^0}$$

$$\times \sum_{j_{\ell+1}=1}^{m_{\ell+1}} \lambda_{\ell+1}^{j_{\ell+1}} \sum_{j_{\ell+2}=1}^{m_{\ell+2}} \lambda_{\ell+2}^{j_{\ell+1}} \cdots \sum_{j_\ell=1}^{m_\ell} \lambda_k^{j_k} \sum_{v_{\ell+1}=1}^{\bar{m}_{\ell+1}} \rho_{\ell+1}^{v_{\ell+1}} \sum_{v_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{v_{\ell+2}} \cdots$$

$$\cdots \sum_{v_k=1}^{\bar{m}_k} \rho_k^{v_k}\,\delta_{\ell+1}^k \frac{\partial v^{(v_k)k}\left(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};v_\ell v_{\ell+1}\cdots v_{k-1}},p\right)}{\partial W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};v_\ell v_{\ell+1}\cdots v_{k-1}}} \phi_k^{i_k}(W_k^{\theta_{\ell+1}^{j_{\ell+1}}\theta_{\ell+2}^{j_{\ell+2}}\cdots\theta_k^{j_k};v_\ell v_{\ell+1}\cdots v_{k-1}},p)(1+r)^{-(\ell-k)}$$

$$\div \Big[\sum_{w_{l+1}=1}^{m_{l+1}} \lambda_{l+1}^{w_{l+1}} \sum_{w_{l+2}=1}^{m_{l+2}} \lambda_{l+2}^{w_{l+2}} \cdots \sum_{w_k=1}^{m_k} \lambda_k^{w_k} \sum_{\varpi_{\ell+1}=1}^{\bar{m}_{\ell+1}} \rho_{\ell+1}^{\varpi_{\ell+1}} \sum_{\varpi_{\ell+2}=1}^{\bar{m}_{\ell+2}} \rho_{\ell+2}^{\varpi_{\ell+2}} \cdots$$

$$\cdots \textstyle\sum_{\varpi_h=1}^{\bar{m}_h} \rho_h^{\varpi_h}\, \delta_{\ell+1}^h \dfrac{\partial v^{h(\varpi_h)}\left(W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}},p\right)}{\partial W_h^{\theta_{\ell+1}^{w_{\ell+1}}\theta_{\ell+2}^{w_{\ell+2}}\cdots\theta_h^{w_h};\varpi_\ell\varpi_{\ell+1}\cdots\varpi_{h-1}}} \Big],$$

for $\ell \in \{1,2,\cdots,T\}$, $k \in \{\ell+1,\ell+2,\cdots,T\}$, $i_k \in \{1,2,\cdots,n_k\}$, $h, i_\ell \in \{1,2,\cdots,n_\ell\}$ and $\upsilon_\ell \in \{1,2,\cdots,\bar{m}_\ell\}$. ■

References: *D.W.K. Yeung: Dynamic Consumer Theory – A Premier Treatise with Stochastic Dynamic Slutsky Equations, Nova Science Publishers, New York, 2015.*
D.W.K. Yeung: Random Horizon Stocahstic dynamic Slutsky Equation under Preference Uncertainty, Applied Mathematical Sciences, Vol. 8, 2014, pp.7311-7340.

D. Number Theory

D1. The Number of Embedded Coalitions

Let $N = \{1,2,\cdots,n\}$ be a finite set of n players in a n −person game. The subsets of N are coalitions. A partition $\Lambda$is formed by disjoint non-empty subsets of N representing a way that these n players are joined. Given a partition $\Lambda$and a coalition $S \subset N$, the pair (S,Λ) is called an embedded coalition, that is the coalition S embedded in partition Λ. The Bell (1934) number, denoted by $\beta(n)$, gives the number of partitions in a n −person game. The number of embedded coalitions in a partition is the number of subsets formed in that partition. The total number of embedded coalitions $\Upsilon(n)$ in a n −person game is the sum of the numbers of embedded coalitions in the $\beta(n)$ partitions of N.

The number of embedded coalitions in a n -person game is:

$\Upsilon(1) = \sum_{t=0}^{0}\binom{1}{t} = \binom{1}{0} = 1,$ for $n = 1$;

$\Upsilon(2) = \sum_{t=0}^{1}\binom{2}{t} = \binom{2}{1} + \binom{2}{0} = 3,$ for $n = 2$;

$\Upsilon(n) = \sum_{t=2}^{n-1}\binom{n}{t}(\sum_{k=1}^{t-1}\Upsilon(k)) + \sum_{t=0}^{n-1}\binom{n}{t},$ for $n \geq 3$. ■

References: *D.W.K. Yeung: Recursive Sequences Identifying the Number of Embedded Coalitions, International Game Theory Review, Vol. 10(1), 2008, pp.129-136.*
E. T. Bell [1934] Exponential numbers, American Mathematical Monthly 41, 411-419, 1934.

D2. The Number of Embedded Coalitions where the Position of the Individual Player Counts

The number of embedded coalitions in a n -person game where the position of the individual player counts is:

$\phi(1) = 1\sum_{t=0}^{0}\binom{1}{t} = 1,$ for $n = 1$;

$\phi(2) = 2\sum_{t=0}^{1}\binom{2}{t} = 6,$ for $n = 2$;

$\phi(n) = n!\Big[\ \sum_{t=2}^{n-1}\binom{n}{t}\Big(\sum_{k=1}^{t-1}\frac{\phi(k)}{k!}\Big) + \sum_{t=0}^{n-1}\binom{n}{t}\ \Big]$, for $n \geq 3$. ■

References: *D.W.K. Yeung, E.L.H. Ku and P.M. Yeung: A Recursive Sequence for the Number of Positioned Partitions, International Journal of Algebra, Vol. 2, 2008, pp.181-185.*

E. Probability Density Functions

E1. Stationary Probability Density Function of the Stochastic Lotka-Volterra-Yeung Food-chain

The function

$$\varphi(N) = w\, exp\left(\frac{-2}{\sigma^2} b_1 N_1 + b_1 \right) exp\left(\frac{-2}{\sigma^2}\frac{b_1}{b_2} c_1 N_2 + c_2 \right)$$
$$\times \prod_{i=3}^{n} exp\left(\frac{-2}{\sigma^2}\frac{b_1}{b_2}\prod_{j=3}^{i}\frac{c_{j-2}}{c_j} c_{i-1}\, N_i + c_{i-1}\right)$$
$$\times N_1^{\frac{2}{\sigma^2}z_1} N_2^{\frac{2\, b_1}{\sigma^2 b_2}z_2} \prod_{i=3}^{n} N_i^{\frac{2\, b_1}{\sigma^2 b_2}\prod_{j=3}^{i}\frac{c_{j-2}}{b_j}z_i} \left(\frac{1}{\prod_{i=1}^{n} N_i} \right)$$

gives the stationary probability density of species $N_1,\ N_2,\ \dots,\ N_n$ of the stochastic Lotka-Volterra-Yeung food-chain:

$$dN_1(t) = \left[a_1 - b_1 N_1(t) - c_1\big(N_2(t)\big)\right] N_1(t)dt + \sigma N_1(t)dz(t),$$
$$dN_i(t) = \left[-a_i + b_i\big(N_{i-1}(t)\big) - c_i N_{i+1}(t)\right] N_i(t)dt,$$
$$\text{for } i = 2,3,\cdots,n-1,$$
$$dN_n(t) = [-a_n + b_n N_{n-1}(t)] N_n(t)dt,$$

where $N_i(t)$ is the population level of the species in the i^{th} trophic level at time t; $z(t)$ is a standard Wiener process, σ, a_i and b_i for $i \in [1,2,3,\cdots,n]$ and c_i for $i \in [1,2,3,\cdots,n-1]$ are positive constant; and $z_1,\ z_2,\ \cdots,\ z_n$ satisfies:

$$z_1 + z_2 = a_1 - \frac{1}{2}\sigma^2,$$
$$-\frac{b_2}{b_1} z_1 + z_3 = -a_2,$$
$$-\frac{b_3}{c_1} z_2 + z_4 = -a_3,$$
$$-\frac{b_4}{c_2} z_3 + z_5 = -a_4,$$
$$\vdots \qquad\qquad \vdots$$
$$-\frac{b_{n-1}}{c_{n-3}} z_{n-2} + z_n = -a_{n-1}.$$

References: *D.W.K. Yeung: Exact Solutions for Steady-State Probability Distribution of a Simple Stochastic Lotka Volterra Food Chain. Stochastic Analysis and Applications, Vol. VI, 1988, pp. 103-116.*
D.W.K. Yeung: Optimal Management of Replenishable Resources in a Predator-Prey System with Randomly Fluctuating Population. Mathematical Biosciences, Vol. 78, 1986, pp. 91-105.

E2. Stationary Probability Density Function of Generalized Stochastic Food-chain of the Lotka-Volterra-Yeung Type

The function

$$\psi(N) = m \prod_{i=1}^{n\prod[[2A_i\, ln\, N_i - 2F_i(ln\, N_i) + 2F_i(0)]/\sigma^2\,]} \frac{1}{N_i} exp$$

gives the stationary probability density of species $N_1,\ N_2,\ \dots,\ N_n$ of the generalized Lotka-Volterra-Yeung type of stochastic food-chain:

$$dN_1(t) = \left[\alpha_1 N_1(t) - b_1 N_1(t) f_1\big(N_1(t)\big) - v_1 N_1(t) f_2\big(N_2(t)\big)\right]dt + \sigma\sqrt{b_1} N_1(t)dz(t),$$
$$dN_2(t) = [\alpha_2 N_2(t) - b_2 N_2(t) f_2(N_2(t)) - v_2 N_2(t) f_3(N_3(t))$$
$$+ v_1 N_2(t) f_1(N_1(t))]dt + \sigma\sqrt{b_2} N_2(t)dz(t),$$
$$dN_3(t) = [\alpha_3 N_3(t) - b_3 N_3(t) f_3(N_3(t)) - v_3 N_3(t) f_4(N_4(t))$$
$$+ v_2 N_3(t) f_2(N_2(t))]dt + \sigma\sqrt{b_3} N_3(t)dz(t),$$
$$\vdots \qquad\qquad \vdots$$
$$dN_{n-1}(t) = [\alpha_{n-1} N_{n-1}(t) - b_{n-1} N_{n-1}(t) f_{n-1}(N_{n-1}(t)) - v_{n-1} N_{n-1}(t) f_n(N_n(t))$$
$$+ v_{n-2} N_{n-1}(t) f_{n-2}(N_{n-2}(t))]dt + \sigma\sqrt{b_{n-1}} N_{n-1}(t)dz(t),$$
$$dN_n(t) = \left[\alpha_n N_n(t) - b_n N_n(t) f_n\big(N_n(t)\big) + v_{n-1} N_n(t) f_{n-1}\big(N_{n-1}(t)\big)\right]dt$$
$$+ \sigma\sqrt{b_n} N_n(t)dz(t),$$

where $N_i(t)$ is the population level of the species in the i^{th} trophic level at time t;
v_i for $i \in [1,2,\cdots,n-1]$ are positive constants, b_1 is positive and b_i for $i \in [2,3,\cdots,n]$ are nonnegative constants;
$\alpha_1 > 0$, and α_i for $i \in [2,3,\cdots,n]$ are constants with α_i being positive when $b_i > 0$ and negative when $b_i = 0$;

$f_i(0)=0$ and $f_i(N_i)>0$ for positive values of N_i, and $f_i(N_i)$ is a continuous differentiable and monotonically increasing in N_i, and $f_i(e^s)$ is an integrable function yielding $\int_0^{x_i} f_i(e^s)ds = F_i(x_i) - F_i(0)$, for $i = 1, 2, \dots, n$;
and
$A_1, A_2, \cdots, A_n$ satisfies

$$
\begin{aligned}
& b_1 A_1 + v_1 A_2 = \alpha_1 - \tfrac{1}{2} b_1 \sigma^2, \\
& -v_1 A_1 + b_2 A_2 + v_2 A_3 = \alpha_2 - \tfrac{1}{2} b_2 \sigma^2, \\
& -v_2 A_2 + b_3 A_3 + v_3 A_4 = \alpha_3 - \tfrac{1}{2} b_3 \sigma^2, \\
& \qquad \vdots \qquad\qquad \vdots \\
& -v_{n-2} A_{n-2} + b_{n-1} A_{n-1} + v_{n-1} A_n = \alpha_{n-1} - \tfrac{1}{2} b_{n-1} \sigma^2, \\
& -v_{n-1} A_{n-1} + b_n A_n = \alpha_n - \tfrac{1}{2} b_n \sigma^2 \omega_n.
\end{aligned}
$$

References: *D.W.K. Yeung: An Explicit Density Function for a Generalized Stochastic Food-chain of the Lotka-Volterra-Yeung Type, Stochastic Analysis and Applications, Vol. 27, 2009, pp.16-23.*